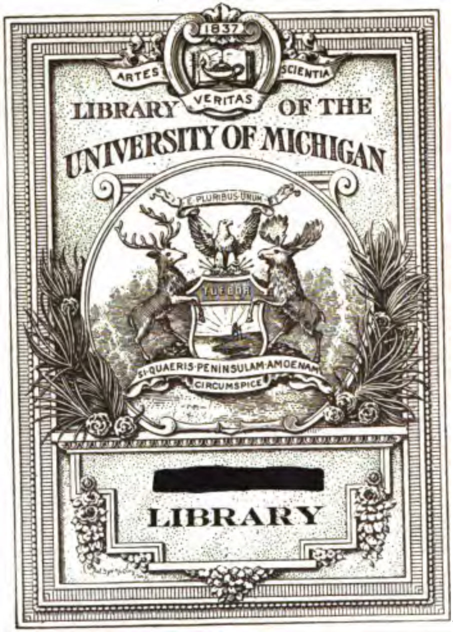
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ELEMENTARY MECHANICS

LESSONS

IN

ELEMENTARY MECHANICS

INTRODUCTORY to the STUDY of PHYSICAL SCIENCE

DESIGNED FOR THE USE OF SCHOOLS
AND OF CANDIDATES FOR THE LONDON MATRICULATION
AND OTHER EXAMINATIONS

WITH NUMEROUS EXERCISES

BY

SIR PHILIP MAGNUS

AUTHOR OF CLASS-BOOK ON 'HYDROSTATICS AND PNEUMATICS' ETC.

NEW EDITION, REWRITTEN AND ENLARGED IN 1892

FORTIETH THOUSAND

LONGMANS, GREEN, AND CO.
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1896

PREFACE

SINCE the publication in January, 1875, of the first edition of my 'Lessons in Elementary Mechanics' a distinct advance has been made in the teaching of the subject, and I have, therefore, thought it necessary to rewrite the book, in order that it may continue to occupy the position it still holds in schools and science classes.

The present edition contains such new matter as experience has shown may be included in a first year's course of instruction. I have made some changes in the definitions as given in previous editions, and greater exactness will be found in the use of several of the terms employed. In many of the most recent text-books there is still some looseness in the use of the words *weight*, *power*, *tension*, &c., and from this fault examination questions are not always free. On the subject of units and change of units I have added one or two sections. As regards the measure of force,

whilst carefully distinguishing between absolute and gravitation units, I have employed both freely, adopting in some instances the C.G.S. system, and in others our ordinary units of length and mass. In this edition, too, I have further aimed at making the teaching of mechanics serve as a basis for the study of other branches of physical science.

The book is intended now, as it was when first written, for school use, and I have seen no reason to alter the arrangement, which, although somewhat novel then, has since been generally adopted. In writing a school text-book the author does not aim so much at obtaining results by the shortest possible methods as at bringing out the educational value of the subject he is treating; and to this end reference is frequently made to first principles rather than to formulæ, and facts and propositions are carefully illustrated and explained, and where necessary are presented to the pupil from more points of view than one.

During the past seventeen years I have received from schoolmasters and other teachers who have used the book numerous suggestions for its improvement, many of which I have adopted. For the advice thus freely offered I take this opportunity of acknowledging my obligation. To Professor R. M. Walmsley I am particularly indebted

for his valuable help in revising, in the first instance, the text of the old edition, and for his assistance in correcting the proofs; and I have also to thank the author of the solutions to the Exercises and Examination Questions, which are separately published, for his careful revision of the Answers at the end of this volume.

The Examination Questions comprise all those set during the last twenty years at the matriculation of the London University, and several from the papers of the Science and Art Department. It is hoped, therefore, that the book will continue to prove serviceable to candidates for these and other examinations.

As showing the original aim and scope of this little work I append an extract from the preface to the first edition.

P. M.

CITY AND GUILDS OF LONDON INSTITUTE,
Exhibition Road, April 1892.

FROM THE PREFACE TO THE FIRST EDITION

In these Lessons, which are intended for the use of those who have had no previous instruction in the subject, I have endeavoured to bring into prominence the leading principles of Mechanics, and to exemplify them by simple illustrations; and with the view of showing the connection between this science and other branches of Physics, some few pages have been set apart to a brief exposition of the Doctrine of Energy.

In arranging the contents of this volume I have deviated considerably from the plan usually adopted, and have been guided by the general principle that the idea of Motion is more elementary than that of Force, and that two Forces, at least, must combine to produce Equilibrium. In accordance with this view the subject of Statics has been made to depend on the laws of Dynamics, and these are preceded by a discussion of some of the simplest principles of Motion. I cannot help thinking that the theory of Equilibrium occupies too prominent a position in many of our Text-books, and that the student obtains, in the problems of Statics, a very inadequate idea of

Force and of its modes of expression. In the present arrangement I have followed that order which appears to me to be most logical, and which experience in teaching has shown to be practically advantageous.

The book contains that amount of matter which a pupil may be expected to acquire in a first year's course of instruction in the subject. It is divided into a number of sections, which may serve as separate lessons, and should be studied in the order in which they occur. All important propositions are illustrated by numerical examples, worked out in the text, and the lessons are furnished with exercises progressively arranged.

. . . .

In writing these Lessons I have had in view the want, which is very generally felt, of a School Text-book sufficiently elementary to be placed in the hands of a beginner, and yet affording a trustworthy basis for the subsequent work of the student. . . . . . .

P. M.

January 1875.

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ELEMENTARY MECHANICS

INTRODUCTION

§ 1. **Motion.**—Our earliest observations must have shown us that some things are moving and that others appear to be at rest. We know what motion means when we watch the rising of the sun, the passage of a bird through the air, the waving of the trees in the wind, or the rushing of the waves to the sea-shore. Every variety of matter seems to be endowed with the faculty of movement. The stone falls to the ground, the flower opens its petals to the sun, and living creatures of every size and shape move in endless ways.

§ 2. **Rest.**—We are equally familiar with bodies which appear to be in a state of rest. Nothing seems more immovable than the earth on which we stand. The various things we see around us—the books that lie upon our shelves, the pictures that hang upon our walls—are all apparently

at rest, and we expect them to remain so unless they happen to be disturbed by some external cause. A little thought will show us that this state of rest is not as simple as it seems. Let us suppose that we are travelling in a railway-carriage, and that another train is moving in the same direction on adjoining rails. After a time it overtakes us, and then the two trains move on side by side with equal speed. In this case all sense of motion will be lost; the train at which we are looking and the carriage in which we sit will equally appear to be at rest. This simple illustration is sufficient to make us see that objects may be moving when we suppose them to be stationary, and that the evidence of our senses cannot wholly be trusted. Now the earth on which we stand is in the position of the second train. It is moving round the sun with a considerable velocity; but, as we are moving with it and at the same rate, it appears to us to be at rest.

Let us consider, further, the condition of those bodies which, although *absolutely* moving with the earth, are, *relatively* to us, at rest. Take the picture hanging on the wall. The picture is suspended by cords which hang over a nail. If these cords were to break, or the nail were to give way, the picture, we know very well, would at once fall to the ground. It appears, therefore, that the

picture, although at rest, is really *tending* to fall, and is only prevented from obeying its natural tendency by the cords and nail that hold it back. What is true of the picture is true of all things that are in any way supported. Each article of furniture in this room would fall through the floor if the floor were not strong enough to support it. There is a vessel on the table filled with water, and in the side of the vessel is a cork. The water appears to be motionless. Remove the cork, and the water immediately begins to flow out. This water, then, is endowed with a tendency to motion, which the sides of the vessel resist. The air of the room is seldom still. But suppose for a moment that there is no kind of draught. Let a window or fire-place be opened, let the air be freed in some direction from restraint, and it will at once obey its tendency and begin to move. In these examples no reference has been made to the cause or causes that are supposed to produce the several movements indicated. But the causes themselves do not come within the range of our observing faculties. All that observation teaches us is, that bodies tend to move.

§ 3. **Molecular Motion.**—If we examine matter more minutely, we shall find that it consists of very small particles, which cannot be broken into smaller particles by any mechanical

means. These molecules are so minute that we cannot, even with our finest microscopes, individually distinguish them ; but there are, nevertheless, other methods by which we can approximately measure their size. Now among these particles movements are constantly taking place, which, because of the minuteness of the moving bodies, are separately hidden from the eye, but the total result of which can be discerned. If we dip the end of a fine glass tube into water, we shall observe that the water rises in the tube, and that the finer the tube the greater will be the height to which it rises. In this case the motion of the water is due to movements amongst the molecules at the surface, where the water and glass meet. Again, we have all noticed, perhaps, that on a line of rails a certain space is left between the separate pieces at the points where they are joined together. This space is left because it is found that the length of the rails increases in hot weather and decreases in cold weather ; and if the line of rails consisted of one continuous bar of iron fixed at its two extremities it would become bent and twisted, in order to find room for its expansion. If we put a liquid into a glass vessel, and place the vessel over the flame of a spirit lamp, we shall very soon observe that the liquid is rising in the vessel ; and after a time it will begin to boil, and its particles will be violently agitated. We know how seldom perfect stillness

prevails in the atmosphere. The wind is always blowing somewhere. Now the motion of the air is caused, directly or indirectly, by the sun's heat, and the sun's heat is ever varying in intensity. These examples serve to show that heat changes are accompanied by motion; but this motion usually takes place among the particles themselves of which the body consists. The body, as a whole, does not move from place to place; but with every variation in its temperature there is a corresponding movement among the particles which compose it.

We can take another illustration. Most persons know what happens if a stick of sealing-wax be rubbed on flannel and then held over some scraps of paper. The pieces of paper will at once begin to move towards the wax, and may be made to stand on end under its influence. They are electrified, and in that condition they tend to move. Similarly, if the end of a bar magnet be brought over some small iron tacks on the table, the tacks will move towards the magnet, and if the magnet be a strong one they will even fly up to it. They are *magnetised* by the bar, and move towards it. Now we cannot say to what extent the molecules of all bodies are thus influenced; for where opposing tendencies to motion exist, which act against one another, no visible effect will be produced.

We have hitherto considered inanimate matter. Let us now see what happens in the animal and vegetable world. A plant or animal may be said to differ from a piece of lifeless matter by its *growth and decay*. Now growth implies a continuous change in the molecules of which a body consists. A living organism cannot preserve its old particles and at the same time acquire new. It increases by the decay of old and the substitution of new matter. In this respect animate bodies increase much in the same way as a merchant's capital. A capitalist cannot grow rich by hoarding : on the contrary, he must become daily poorer, for how parsimonious soever he may be, he must consume a portion of what he possesses to support life. It is only by spending money, by buying and selling, by constantly exchanging capital and allowing it to be used by labourers as food, that capital can increase. The same is true of living tissue. In growing it undergoes continuous decay, and the decay is continuously repaired. When the process of reparation does not proceed as rapidly as that of decay, the plant begins to fade and the animal to die. In this continuous decay and reproduction we have a further example of motion among the molecules of bodies.

We thus see that bodies themselves and their

molecules are constantly in motion or tending to move relatively to one another; that absolute rest nowhere exists; and that what we call rest, which is really rest relatively to us, can be resolved into counteracted tendencies to motion. As motion is thus universally present, we are sensible of what it is, without being able to define it. It does not admit of explanation; for there is no condition in which matter exists that is simpler or more elementary.

It is important to grasp this proposition at the outset, for there were wise men of old who, starting with the state of rest as the simplest condition of matter, endeavoured to explain motion, and they were met at once by this difficulty: that a body in motion is no sooner in its place than it is out of it, and that the idea of change involves the conception of a thing existing and not existing—i.e. of its being in a place and not being in that place at the same time. It is better, therefore, to consult nature, by which we see that all things are constantly moving, or tending to move, that there is no matter without motion, and that what we call rest is a strained condition resulting from tendencies to motion counteracted.

§ 4. **Varieties of Motion.** (*a*) *Translation.*—There are different kinds of motion. Let us see

what they are. If a body moves from one place to another it is said to undergo *translation.* It may move in a straight line or in a curve. The run of a curling-iron along the ice and the fall of a stone in a well are examples of motion in a straight line. The flight of an arrow and the course of the planets illustrate what is meant by curvilinear motion.

(*b*) *Rotation.*—When a body moves about a fixed point or axis, round which the particles describe concentric circles, the body is said to *rotate.* Thus a top rotates on its axis, a wheel on its axle, a door on its hinges.

It frequently happens that several motions are combined. A body tending to move in two different directions may be found to move in a straight line between them, or to assume a curvilinear motion as the result of these two tendencies. Thus the path of a cricket-ball through the air is a curved line, which is the joint effect of the tendency of the ball to move in the direction in which it was struck, and of its tendency to fall to the earth. Further, the motion of rotation is frequently combined with that of translation. This occurs when a wheel rolls along the ground, or when a billiard-ball runs along the table. The motion of the earth round the sun is the result of two tendencies to motion in different directions,

producing a curve called the orbit, and also of a rotation about a fixed axis passing through the poles. When a body is under the influence of opposing tendencies to motion which exactly counterbalance one another it is said to be in equilibrium.

(c) *Wave Motion.*—There is another kind of motion, to which the name *undulatory* has been applied. It exists under a variety of forms, but may roughly be described as the movement of a particle to and from a particular point. This displacement of the particle is often called its excursion, and is, in all cases, very small. When a series of neighbouring particles undergo successively this kind of to-and-fro motion a *wave* is said to be produced. The peculiarity of wave-motion is that, although the particles never move beyond the limits of an excursion, they appear to undergo translation. If we fix our eyes on a piece of wood floating on a sea-wave, we shall observe that, whilst the wave appears to approach ever nearer to the shore, the piece of wood maintains its position, rising and falling continuously. The apparent motion of translation is the result of the up-and-down movement of each successive particle in its own place. This kind of motion is further illustrated when the wind sweeps over a field of corn and bends the several ears in succession. In this case it is evident that each stalk of corn

cannot move out of its own place, and yet the eye can follow the wave as it passes from one end of the field to the other. In an undulation, the particular motion of each particle is well illustrated by the movement of the bob of a pendulum, which vibrates to and from the lowest point in its path.

§ 5. **Physics Defined.**—The science of Physics embraces the consideration of bodies and molecules under every variety of motion, and is subdivided according to the particular effect the several kinds of motion produce. Thus the passage of a bird through the air is a case of locomotion ; the resistance of a heavy body tending to fall to the earth produces pressure. Certain motions produce heat, others give rise to the phenomena of sound, light, electricity, and magnetism.

Physical science is divided into two main branches, according as the motion to be considered is the motion of a body as a whole or of the undulations of the molecules of which it consists. It has been suggested to call the one branch of the subject 'Molar Physics,' as treating of *motion in mass*, and the other 'Molecular Physics,' as treating of the *motion of molecules*. The flight of a rifle-bullet, the blow of a cricket-bat, the ascent of a balloon are questions of Molar Physics ; whilst problems concerning temperature, musical

notes, and electric light belong to the other branch of the subject.

§ 6. **Mechanics.**—The term Mechanics was used by Newton to denote the science of machines and the art of making them, but is now generally employed to embrace the science of the motion and equilibrium of bodies. It involves the consideration of matter in its three forms—solid, liquid, and gaseous. The subject at once divides itself into two branches, the one called *Kinematics* [1] and the other *Dynamics*.[2] Kinematics, or the science of motion, treats of the principles of motion apart from the consideration of the quantity of the matter moved or of the causes or effects of the motion. Dynamics, or the science of force, treats of the motion of bodies in connection with the quantity of matter moved and the measure of the force producing or called into action by the motion. The subject of dynamics is usually divided into two branches, *Kinetics* and *Statics*,[3] according as the action of the forces results in motion or rest. Kinetics is that branch of dynamics which is concerned mainly with the measurement and effect of forces that produce or change the motion of bodies; whilst in statics the action of forces in maintaining

[1] Greek *κινέω*, I move. [2] Greek *δύναμις*, force.
[3] Latin *sto*, I stand.

equilibrium or preventing motion or change of motion is investigated. The above divisions may be thus exhibited :—

Mechanics

Kinematics — Dynamics

Kinetics — Statics

KINEMATICS—MOTION

CHAPTER I

MEASUREMENT OF MOTION

I. *Uniform Motion*

§ 7. **Speed and Velocity.**—The first question we have to determine is how motion may be measured and numerically represented. Now we measure all things by their effects, and the visible effect of motion is change of place. But connected with the idea of motion as change of place is always that of *quickness* or *slowness*; and when we consider motion with reference to *time* we obtain the idea of *speed*. Thus speed, or rate of motion, is measured by the amount of change of place, or displacement in a given time. But our conception of motion involves something more. A body may be moving at a certain rate in one direction or another, and we must necessarily think of its moving in some direction. When we state

not only the rate of motion, or speed, but also the *direction* in which the change takes place, we have completely specified the motion ; and to this idea involving displacement, time, and direction, we give the name *velocity*. The direction of motion has, of course, to be specified by reference to known directions, such as N.E., S.W., vertically upwards, and so forth.

Motion may be uniform or variable. When uniform, equal distances are traversed in equal times, and the speed is *constant*. When the speed changes and equal distances are not traversed in equal times, the motion is said to be *variable*.

§ 8. **Units and Change of Units.**—In measuring all physical quantities certain fundamental units are arbitrarily adopted as standards for comparison. In dealing with speed or velocity units of distance and time are required. The distance of one point from another, or the displacement of a point, is denoted by length, and length is measured in certain units, such as a mile, a yard, or a metre. The unit of time may be a year, a month, or a day ; but in physical investigations it is everywhere the mean solar second, or, briefly, a second. The unit of length is usually one foot or one *centimetre*.

The centimetre is the hundredth part of a metre, and is equal to 0·39370432 inch, or nearly

two-fifths of an inch. The metre is also divided into tenths, called decimetres, a decimetre being equal to nearly four inches. Taking the foot as the unit of length, the speed or velocity of a moving body is measured by the number of feet traversed in one second; and a body is said to be moving with a unit of velocity when it moves through one foot in one second, or through one centimetre in one second. The unit of speed may be briefly indicated as a *foot per second* or a *centimetre per second.*

If the units of measurement are changed the number denoting the speed will also be changed. Thus a speed of 12 feet per second is the same as 720 feet per minute, or 4 yards per second, or 240 yards per minute.

It is important to notice that the *increase* in the unit of *length diminishes* the number expressing the speed, whereas the *increase* in the unit of *time increases* the number. Whenever in the following pages a velocity or speed is indicated by a simple number the unit implied will be a foot per second. Thus a speed of 12 will mean a speed of 12 feet per second. The units may often be conveniently expressed in brackets. Thus: a speed of 10 (metres, minutes) means a speed of 10 metres per minute, and equals $10 \times \frac{100}{60} = \frac{50}{3}$ centimetres per second.

When the velocity is variable, it is, at any moment, measured by the distance through which the moving body would pass in one second, if it were to continue to move throughout that second with the velocity which it had at that particular moment.

§ 9. **Uniform Velocity.**—If u be the uniform velocity of a moving body, *u equals the number of feet which a body traverses in one second,* and

$2u$ is the number of feet traversed in 2 seconds.					
$3u$	,,	,,	,,	3	,,
.	.	.	.	.	.
tu	,,	,,	,,	t	,,

If $s =$ distance traversed in t seconds, then $s = tu$. This is the fundamental proposition of uniform motion.

Suppose a body to be rotating about a fixed axis, like the sail of a windmill, and that it sweeps out an angle the measure of which is ω in one second, then ω is said to be the *angular* velocity of the body, and $t\omega$ will be the angle described in t seconds. If ω is expressed by *radians* (i.e. circular measure, or arc $\div$ radius), the actual distance traversed in one second by a point r feet from the axis will be ωr, which is, therefore, the speed of the point. Substituting this value of u in our

fundamental equation $s = u\,t$, we get $s = \omega\,r\,t$ for the distance traversed in t seconds by a point r feet from an axis, and moving round that axis with a uniform angular velocity ω.

EXERCISES I

1. A body moving with a uniform speed travels over 276 feet in $2\frac{1}{4}$ seconds; what is its rate of motion?
2. If a body is moving in a given direction with a uniform velocity of 30 centimetres per second, how long will it take to traverse a distance of 12 metres?
3. A train passes a certain point at the rate of 45 miles an hour; express this velocity in feet per second.
4. A man walks at the rate of 6,000 metres per hour; express the speed in centimetres per second.
5. A wheel, radius 50 centimetres, revolves 50 times in $2\frac{1}{2}$ seconds; find the actual distance traversed in one second by a point on the circumference.
6. With what velocity in feet per second does a man walk who traverses 3 miles in a half-hour?
7. How long would a train take to go 240 miles with a velocity of 22 feet per second?
8. Express in metres and minutes a velocity of $3\frac{1}{2}$ miles an hour.
9. Which is the greater velocity, 950 centimetres per minute or 580 metres per hour, and by how much?
10. The measure of a certain velocity is 11 when the units are a foot and a second; find the measure of the same velocity when the units are (1) a mile and an hour, (2) a yard and a day, (3) an inch and a minute.

II. *Variable Motion—Acceleration*

§ 10. **Variable Velocity.**—The velocity of a body may change in magnitude or in direction or both. When a stone falls to the ground the velocity changes in magnitude; when a ball is thrown from the hand the velocity changes in direction. The velocity of a body may increase or decrease, uniformly or variably. If it increase or decrease uniformly, the motion is said to be uniformly accelerated. The rate of change of the velocity is called the *acceleration.* The acceleration may be positive or negative. It is said to be positive when the velocity *increases*, negative when it *decreases.* If the velocity change, but not uniformly, the acceleration is said to be variable. As problems connected with variable acceleration are very complicated, requiring for their solution the higher parts of mathematics, we shall consider, in the following pages, uniform acceleration only.

§ 11. **Measure of Acceleration.** — *Uniform acceleration* is measured by the change of the velocity in a unit of time, that is, by the number of units of velocity by which the velocity is increased or decreased in a unit of time. Thus, suppose a body is found to be moving, at the beginning of three successive seconds, with the velocities

10, 15, and 20 respectively, the body is said to be moving with a uniform positive acceleration of 5. So, too, if a body started with a velocity of 60 feet per second, and at the end of the first second was moving with a velocity of 50 only, and at the end of the next second with a velocity of 40, the body would be said to be moving with a negative acceleration of 10. Expressed more fully, an acceleration of 5, the units being feet and seconds, means an increase of velocity each second of 5 feet per second. The student must carefully consider the import of the final words 'per second,' for the velocity might increase at the rate of 5 feet per second each minute, which would really be an acceleration equal to $\frac{1}{60}$ of the former acceleration. The repetition of the unit of time is necessary in all questions of acceleration. A man's wages might increase every year 5*l.* a year; they might also increase every month 5*l.* a year, in which case the annual increase would be 12×5, or 60*l.* a year.

When we use a number only to represent acceleration it will be understood that the units are a foot and a second; when we wish to indicate the units they should be given in brackets. Thus an acceleration of 7 (centimetres, seconds) means that the velocity increases by 7 centimetres per second per second; an acceleration of -8 (yards, minutes)

means that the velocity *decreases* by 8 yards per minute per minute.

If, then, a represent the acceleration of a moving body,

a is the velocity gained or lost in 1 second ;

and, if the acceleration is supposed to be uniform,

$2a$ is the velocity gained or lost in 2 seconds,					
$3a$	"	"	"	3	"
and ta	"	"	"	t	"

If we call v the increase or decrease of the velocity in t seconds, then $v = at$.

§ 12. **Examples.**—(1) A body starting from rest has been moving with a uniform acceleration for 5 minutes, and has acquired a velocity of 30 miles an hour ; what is the measure of the acceleration ?

$$\text{Here the vel.} = \frac{30 \times 1760 \times 3}{60 \times 60} = 44 \text{ feet per second,}$$

$$\text{and } v = at \therefore 44 = 5 \times 60 \times a \therefore a = \frac{11}{75}.$$

(2) If a body move from rest with a uniform acceleration of $\frac{2}{3}$ (cents., secs.), how long must it be moving to acquire a velocity of 6,000 metres per hour ? Here

$$\text{vel.} = \frac{6000 \times 100}{60 \times 60} \text{ cents. per sec.} = \frac{500}{3} \text{ cents. per sec.,}$$

$$\text{and } a = \frac{2}{3} \therefore v = \frac{500}{3} = \frac{2}{3}t \therefore t = 250 \text{ seconds.}$$

§ 13. **Initial, Final, and Average Velocity.**—Suppose the velocity of a body moving with a uniform acceleration a to increase from u to v in t seconds, then u is called the *initial* and v the *final* velocity for that interval of time, and the increase of velocity is $v - u$.

It follows, therefore, from § 11, that the gain of velocity $v - u = a\,t$, or

$$a = \frac{v - u}{t}.$$

Since, too, the velocity increases uniformly with the time, the *average* velocity throughout the time t equals the mean of the initial and final velocities; i.e.

$$\text{the average velocity} = \tfrac{1}{2}(u + v).$$

Suppose a train passes a station with a velocity of 12 miles an hour, which is equal to 352 yards per minute, and that the velocity after 5 minutes is 36 miles an hour, or 1,056 yards per minute; the speed having increased uniformly throughout the whole time, the average velocity during the 5 minutes has been $\frac{1}{2}(352 + 1056)$, or 704 yards per minute.

§ 14. **Distance Traversed.**—To find the distance through which a body passes when it moves with a uniform acceleration is a somewhat difficult problem, requiring higher mathematics than we are supposed to have at our command. Later on,

we shall show how this problem may be solved by a purely geometrical method, but now we shall content ourselves with a very simple explanation of it.

If the speed of a body increases uniformly from u to v in the time t, a little thought will show that the distance traversed during this time must be the same as if the body had moved uniformly for t seconds with the mean speed, i.e. we may suppose the body to have been moving with a uniform speed of $\frac{1}{2}(u + v)$. Hence it follows (§ 9) that the distance traversed in the time t is $\frac{1}{2}(u + v)t$. If we call s this distance

$$s = \frac{1}{2}(u + v)t.$$

Enunciating this proposition generally, we may say :—*The distance traversed in any given time by a body moving with a uniform acceleration equals the distance that it would have traversed if it had been moving throughout the given time with a uniform velocity equal to the mean of its initial and final velocities.*

We are now able to determine the distance passed over in t seconds, when a body moves with a uniform acceleration a.

First. Let the body start from rest. In this case the initial velocity $= 0$, and the velocity after t seconds, that is the final velocity $= at$,

$\therefore$ the mean velocity $= \frac{1}{2}(0 + a\,t) = \frac{a\,t}{2}$; and if a body move for t seconds with a uniform velocity equal to $\frac{1}{2}a\,t$, the distance traversed is $t \times \frac{1}{2}a\,t$ (§ 9),

$$\therefore s = \tfrac{1}{2}a\,t^2.$$

Secondly. Suppose the velocity at starting to be u, then t seconds afterwards the velocity will be u + the *increment* or gain of velocity, i.e. $u + a\,t$, and the mean velocity will be

$$\tfrac{1}{2}(u + \overline{u + a\,t}) = u + \tfrac{1}{2}a\,t\,;$$

$\therefore$ if s is the distance traversed in t seconds,

$$s = u\,t + \tfrac{1}{2}a\,t^2.$$

If the velocity *decreases* uniformly, the acceleration is negative, and $s = u\,t - \frac{1}{2}a\,t^2$. Hence generally

$$s = u\,t \pm \tfrac{1}{2}a\,t^2. \quad \text{Or,}$$

The distance traversed in any given time by a body moving with a uniform velocity and a uniform acceleration is the distance it would have traversed if it had moved with the uniform velocity, increased or decreased by the distance it would have traversed if it had moved from rest with the uniform acceleration.

§ 15. **Examples.**—(1) A body is moving with a velocity of 30 ft. per second, and its velocity is each second increased by 10 ft. per second. Find the distance traversed in 5 seconds.

Initial vel. = 30; final vel. = 30 + 50 = 80

$$\therefore \text{ mean vel. } = \frac{30+80}{2} = 55$$

and distance traversed = 5 × 55 = 275 feet.

(2) Find the acceleration, if a body starting with a velocity of 10 ft. per second passes over 90 ft. in 4 seconds.

Let a = acceleration; initial vel. = 10; final vel. = 10 + 4 a.

$$\therefore \text{ mean vel. } = \tfrac{1}{2}(10 + \overline{10 + 4a}) = 10 + 2a$$

$$\therefore \text{ distance traversed} = 4(10 + 2a) = 40 + 8a = 90 \text{ ft.}$$

$$\therefore a = 6\tfrac{1}{4}.$$

(3) In 10 seconds the velocity of a body increases from 300 centimetres per second to 500 centimetres per second; find the distance traversed, and the acceleration.

Here the initial velocity is 300, the final velocity is 500.

Hence the mean vel. is 400 ∴ the distance required is

400 × 10 = 4000 centimetres, or 40 metres.

To find the acceleration, we want to know the gain of velocity in the given time, and, as the gain is supposed to be the same each second, the acceleration equals the gain of velocity divided by the time.

Hence 500 − 300 = 10 a, or a = 20 (cents., secs.)

(4) What is the distance traversed in any particular second, when a body moves with a uniform acceleration a?

If the body start from rest the mean velocity during the first second is $\frac{1}{2}(0 + a) = \frac{1}{2}a$, and ∴ this is the distance traversed in that second. Similarly the

distance traversed in 2nd second is $\frac{1}{2}(a + 2a) = \frac{3}{2}a$,

" 3rd " $\frac{1}{2}(2a + 3a) = \frac{5}{2}a$,

" 4th " $\frac{1}{2}(3a + 4a) = \frac{7}{2}a$.

Hence, it will be seen, by looking at the numbers 1, 3, 5, 7, that the distance traversed in the t^{th} second equals the t^{th} odd number multiplied by $\frac{a}{2}$

$$\text{or } s = (2t - 1)\frac{a}{2}.$$

So also the distance traversed in the first t seconds of the motion equals the sum of the first t odd numbers multiplied by $\frac{a}{2}$; or $s = \frac{1}{2}at^2$, as before (§ 14).

§ 16. The fundamental formulæ for the solution of problems in rectilinear motion are

$$v - u = at \quad . \quad . \quad (1),$$
$$s = \tfrac{1}{2}(u + v)t \quad . \quad . \quad (2),$$
and
$$s = ut + \tfrac{1}{2}at^2 \quad . \quad . \quad (3);$$

but the student should accustom himself to reason out the solutions from first principles, and to regard the formulæ as symbolic expressions only of the processes of such reasoning. By the use of formulæ, however, we are often enabled to determine implied relations between the quantities which are not explicitly expressed

Thus from (1) and (2), by eliminating t by division, we have

$$\frac{s}{v-u}=\frac{\frac{1}{2}(u+v)}{a}, \text{ or } v^2-u^2=2\,a\,s \quad . \quad . \qquad (4)$$

an expression which connects the distance traversed with the acceleration and initial and final velocities.

A special instance of this formula is when the body starts from rest, in which case $u=0$ and $v^2=2\,a\,s$.

Examples.—(1) Find the acceleration of a body which, starting from rest, acquires a velocity of 500 centimetres per second after having moved through a distance of 10 metres.

$$\text{Here, since } v^2=2\,a\,s,\ a=\frac{v^2}{2s}=\frac{250000}{2000}$$

$$=125 \text{ (centimetres, seconds).}$$

(2) What distance must be passed over by a body, moving with an acceleration 5, so that its velocity may be increased from 10 feet per second to 20 feet per second?

Here, since $v^2-u^2=2\,a\,s$, we have

$$20^2-10^2=2\times5\times s;\ \text{or } s=30 \text{ feet.}$$

(3) A train passes a station with a velocity of 20 miles an hour; find the velocity with which it passes another station 15 miles distant, if the measure of the acceleration is 40 miles per hour per hour.

$$\text{Here } v^2=u^2+2\,a\,s=\overline{20}^2+2\times40\times15=1600$$

$$\therefore v=40 \text{ miles per hour.}$$

§ 17. **Change of Units.**—Suppose a body to be moving with an acceleration 5, the units being a

foot and a second. A velocity of 5 feet per second is equivalent to a velocity of 5×60 feet per minute, and this increase of velocity is gained every *second.* But the gain per minute must be 60 times the gain per second, i.e. the gain per minute is $60 \times 5 \times 60$ feet per minute. Hence an acceleration of 5, the units being a foot and a second, is equivalent to an acceleration of 18,000, the units being a foot and a minute, i.e. if the gain of velocity per second is 5 feet per second, the gain per minute is 18,000 feet per minute.

Again, suppose a body is moving with an acceleration of 10 (metres, minutes), and it is required to express this acceleration in centimetres and seconds.

An acceleration of 10 (metres, minutes) means—

The gain of velocity per *minute* is 10 metres per minute;

$\therefore$ the gain of velocity per *minute* is 1,000 centimetres per minute;

$\therefore$ the gain of velocity per *minute* is $\frac{1000}{60}$ centimetres per second;

$\therefore$ the gain of velocity per *second* is $\frac{1000}{(60)^2}$ centimetres per second.

Or an acceleration of 10 (metres, minutes) is equal to an acceleration of $\frac{5}{18}$ (centimetres, seconds).

Hence we see that the numerical value of a given acceleration varies inversely as the factor changing the unit of length, and directly as the *square* of the factor changing the unit of time.

If we take the formula $s = \frac{1}{2} a t^2$ (§ 14) we have $a = \frac{2s}{t^2}$. Suppose the units of length and time to be changed, so that $s = l s'$ and $t = k t'$, in consequence of which the numerical value of a is changed to a'; then

$$a = \frac{2 l s'}{k^2 t'^2} = \frac{l}{k^2} a';$$

or

$$a' = \frac{k^2}{l} a,$$

which shows that, if the units of length and time are changed, the numerical value of the acceleration in changed units is obtained by multiplying this value by the square of the factor changing unit of time and dividing by the factor changing unit of length. The factors l and k may be integers or fractions, and are obtained by expressing the *new* unit in terms of the *old* unit.

It should be observed that acceleration has the same relation to velocity acquired that velocity, when uniform, has to distance traversed.

$$\text{Since } v = a t \therefore a = \frac{v}{t},$$

$$\text{and since } s = u t \therefore u = \frac{s}{t}.$$

In other words, acceleration is the ratio of the increment of velocity to the time, whilst velocity is the ratio of the displacement to the time.

§ 18. **Examples.**—(1) Express in yards and minutes an acceleration of 18 (miles, hours). Since 1 yard = $\frac{1}{1760}$ mile, and 1 minute = $\frac{1}{60}$ hour, $l = \frac{1}{1760}$ and $k = \frac{1}{60}$, and the acceleration 18 (miles, hours) = acceleration $\frac{18 \times \frac{1}{60} \times \frac{1}{60}}{\frac{1}{1760}} = \frac{18 \times 1760}{60 \times 60} = 8{\cdot}8$ (yards, minutes).

(2) A body moving for 3 seconds with a constant acceleration acquires a velocity of 2,940 centimetres per second; express this acceleration in metres, minutes.

Since $v = at$, $a = 2940 \div 3 = 980$ (cents., secs.), and 1 metre = 100 centimetres, and 1 minute = 60 seconds, $\therefore$ accel. 980 (cents., secs.) = accel. $\frac{980 \times 60 \times 60}{100}$ = accel. 35 280 (metres, minutes).

EXERCISES II

1. In what time will a body moving[1] with an acceleration of 25 acquire a velocity of 1,000?
2. What distance will a body traverse in one-tenth of a minute, if the increase of the velocity per minute is 160 yards per minute?
3. With what velocity must a body start, if its velocity be retarded each second 10 feet per second and it come to rest in 12 seconds?
4. In how many seconds will a body describe 1,400 feet, moving from rest with an acceleration of 7?

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[1] In the examples the body is supposed to start from rest, and the units of length and time are taken as a foot and a second, unless otherwise stated.

5. Through what distance will a body move in 4 seconds with an acceleration of 32·2 ?
6. A body moving from rest with a uniform acceleration describes 90 feet in the 5th second of its motion; find the acceleration and velocity after 10 seconds.
7. What is the velocity of a particle which moving with an acceleration of 20 has traversed 1,000 feet ?
8. A body moves with an acceleration of 9·8 (metres, seconds); find the distance traversed in 5 seconds.
9. The velocity of a body changes from 90 centimetres per second to 50 centimetres per second whilst it travels over 3 metres; find the acceleration.
10. Express an acceleration of 32·2 in inches and minutes.
11. Find the distance traversed in the 20th second by a body moving with an acceleration of 1,000 (metres, hours).
12. Find the distance traversed in $3\frac{1}{4}$ seconds by a body moving with an acceleration of 386·4 (inches, seconds).
13. With what acceleration must a body move that, starting from rest, it may travel over 30 miles in 30 minutes ? Express the acceleration in feet, seconds; and in miles, hours.
14. A body is observed to move over 45 feet and 55 feet in 2 successive seconds; find the distance it would traverse in the 20th second.
15. With what velocity is a body moving after 4 seconds if its acceleration is 10 ?
16. The velocity of a body increases every minute at the rate of 360 yards per minute. Express this acceleration, taking a foot and a second as units of length and time. Find the distance traversed from rest in 20 seconds.
17. What velocity must a body have so that, if its velocity be retarded each second 10 feet per second, it may move over 45 feet before coming to rest ?
18. What velocity will be gained by a particle that moves for 5 seconds with an acceleration of 12 ?

III. *Geometrical Representation of Motion*

§ 19. **Curve of Speed.**—In § 14 we have shown how the distance traversed by a body in a given time, moving with a uniform velocity, may be calculated. In subsequent paragraphs we have made certain assumptions from which we have deduced corresponding expressions for variable motions. But, in order that these may be more rigorously proved, it may be useful to consider how

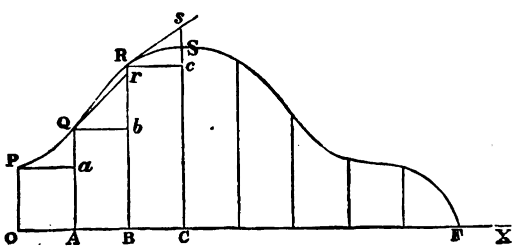

FIG. 1

the quantities with which we have to deal may be geometrically represented.

Let OX (fig. 1) be a line limited towards O, unlimited towards X, on which units of length correspond to units of time; so that, if the lengths OA, AB, BC be equal to one another, and OA represent one second, OB would represent two seconds, and so on. Now suppose we commence to count our time from O, and let the speed with

which a body is moving at any particular time be represented by a vertical drawn through one of the points on $O X$ which corresponds to that time. This should consist of as many units of length as there are units of speed in the velocity to be represented. Thus, if $O P$ represents the speed at the time O, $A Q$ the speed at the time $O A$, and $B R$ the speed at the time $O C$, and so on, then the lines $O P$, $A Q$, $B R$, . . . indicate the number of feet per second with which the body is moving at the times indicated by the points O, A, B, C. . . . In the same way, if all the corresponding verticals be drawn for the moments of time intermediate between O, A, B . . . and their extremities be joined, the line $P Q R S F$ is called the *curve of speed.*

It must not be supposed that the curve of speed is the same thing as the path of a body. Motion might take place in a straight line, and yet the speed-curve might be represented as in fig. 1. The curve of speed is merely a graphic representation of the increase and decrease in the rate of motion at successive intervals of time.

§ 20. **Acceleration.**—If we draw the lines $P a$, $Q b$ parallel to the line $O X$, $Q a$ and $R b$ will respectively represent the increase of the velocity during the first two seconds; and if this increase were uniform, these lines would be equal, and

would represent the acceleration. When the acceleration varies, it is measured at any point by the velocity, which would be added in a unit of time, if the velocity increased uniformly throughout such time. If $Q\,r$ be a tangent to the curve at Q, $Q\,r$ represents the direction the speed-curve would take if, after the time $O\,A$, the acceleration became uniform and continued so during the second represented by $A\,B$. Thus $r\,b$ measures the acceleration at the particular moment of time indicated by A, and when the velocity acquired is $A\,Q$. Similarly $s\,c$ measures the acceleration at the time B, and in this way we obtain a graphic representation of the velocity and acceleration of the body at any instant of time.

§ 21. **Distance Traversed.**—We have now to show how the distance passed over in any given time may be graphically represented. Let $O\,Z$ (fig. 2) represent any interval of time, $A\,O$ the speed at O, $Y\,Z$ the speed at Z, and $A\,P\,Q\,Y$ the curve of motion, as before. Let $F\,G$ be a very small interval of time τ. Let $F\,P$ be the speed at the beginning of the time τ, $G\,Q$ the speed at the end. Then the distance traversed in the time τ must be greater than the distance that would be traversed if the speed $P\,F$ were uniform throughout the interval, and less than the distance that would be traversed with the uniform speed $Q\,G$.

That is, the true distance must lie between $PF \times FG$ and $QG \times FG$ (since $s = v\,t$); but $PF \times FG =$ the rectangle PG, and $QG \times FG$ = the rectangle QF. Therefore the true distance traversed is represented by a figure the magnitude of which lies between the rectangles PG and QF. Now the whole time OZ is made up of the sum of such intervals as FG, and therefore the whole distance traversed in the time OZ is somewhere

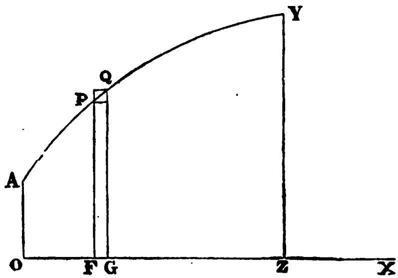

FIG. 2

between the number of units of area in the sum of all the rectangles like PG, and the sum of all the rectangles like QF. But the sum of each of these sets of rectangles approaches nearer and nearer to the area of the whole figure $OAYZ$, as the intervals like FG are made smaller and smaller; and can be made to differ from $OAYZ$ by as small a quantity as ever we please. It thus appears that the distance traversed lies between two

quantities; that each of these quantities becomes ultimately equal to $O A Y Z$, as $F G$ diminishes without limit; and, therefore, that the distance traversed equals the number of units of area in the figure $O A Y Z$. We have thus proved that the distance traversed in any given time may be represented by the number of units of area contained by the two verticals of speed, the included line of time, and the portion of the speed-curve intercepted between these two verticals.

The problem of finding the distance traversed in any time resolves itself into that of finding the area of a curve. In all but the simplest cases a knowledge of higher mathematics is necessary.

§ 22. **Uniform Motion.**—In uniform motion the speed at different periods of time remains the same. The curve becomes, therefore, in this case a

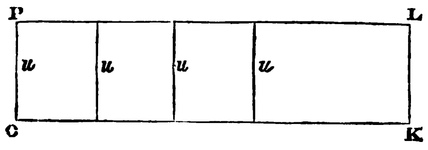

FIG. 3

straight line, parallel to the line of time, and the distance traversed in t seconds equals the number of units of area in the rectangle $O L = O K \times K L = t u$.

§ 23. **To find the distance traversed in t seconds, when a body moves with a uniform acceleration.**

In this case the increments of velocity for successive seconds are constant.

First. Let the body start from rest. Then if $O X$ be the line of time, and $O A$, $A B$ . . represent seconds, and if $P A$ represent the velocity at A, $Q B$ at B, and $R C$ at C; and if $P b$, $Q c$ be drawn parallel

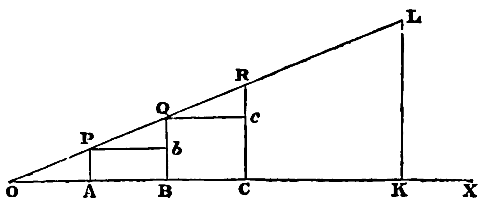

FIG. 4

to $O X$, then $P A = Q b = R c = a$, the acceleration, and $O P Q R$ can be geometrically proved to be a straight line.

Let $L K$ represent the velocity after t seconds, then $O K = t$ and $L K = a t$, and the distance traversed in t seconds equals the area of the triangle $O L K = \frac{1}{2} O K \times K L = \frac{1}{2}$, $t \times a t$, $\therefore s = \frac{1}{2} a t^2$.

Secondly. Let the body start with a given velocity u, and after t seconds acquire a velocity

v, so that $OK = t$ (fig. 5), $OP = u$, and $KQ = v$. Then, since the acceleration is supposed to be uniform, PQ is a straight line, as before, and the distance traversed in t seconds is represented by the area $OPQK$.

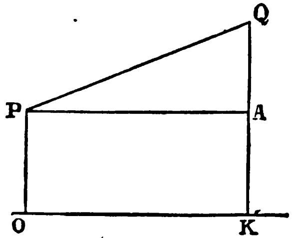

FIG. 5

But the area of the trapezium $OPQK$ is equal to the area of the triangle which has the sum of OP and KQ for a base, and OK for a height. This may be easily proved by producing KQ to a point Z (fig. 6), so that $QZ = OP$, and joining ZO, when the triangle KZO will be found equal to the figure $OPQK$, because the triangles PNO and ZNQ are equal;

$$\therefore \text{ the area } OPQK = \tfrac{1}{2}(OP + KQ) \times OK$$
$$\therefore s = \tfrac{1}{2}(u + v)\,t.$$

If PA be drawn parallel to OK (fig. 6), we have the area $OPQK =$ the rectangle $OPAK +$ the triangle PAQ.

$$\text{But } OPAK = u\,t, \text{ and } PAQ = \tfrac{1}{2}\,a\,t^2,$$
$$\therefore\; s = u\,t + \tfrac{1}{2}\,a\,t^2.$$

§ 24. Since $KQ = KA + AQ$, and $AQ =$ the gain of velocity in t seconds $= a\,t$ (§ 11),

$$\therefore\; v = u + a\,t,$$

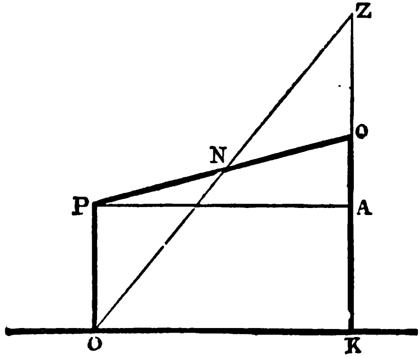

FIG. 6

or $v - u = a\,t$; and by combining this formula with the formula $s = \frac{1}{2}(v + u)\,t$ we have, as before,

$$v^2 - u^2 = 2\,a\,s, \text{ or}$$
$$v^2 = u^2 + 2\,a\,s.$$

If the velocity decrease instead of increasing, the line PQ would slope downwards, and the distance AQ would have to be subtracted from KQ, and the triangle PAQ from the rectangle $OPAK$. In this case the acceleration is negative,

and for a should be substituted $-a$ in the formulæ already established.

EXERCISES III

1. What is the numerical value of an acceleration which in $\frac{1}{3}$ second will produce a speed which will carry a body over 8 feet in every $\frac{1}{2}$ second?
2. Draw a figure representing the distance traversed by a body in 5 seconds moving with a uniform acceleration, whilst its speed increases from 30 units to 75 units.
3. Find the distance traversed by a train whilst its speed is diminished from 60 miles an hour to 30 miles an hour, supposing the rate of retardation to be 10 miles an hour each second.
4. Show graphically the time required to traverse the first, second, and third foot respectively, if a body start from rest and move with a uniform acceleration of 20 (feet, seconds).
5. Find the increase of speed per second if the speed increases from 50 metres per minute to 80 metres per minute whilst the body traverses 500 metres.
6. A body describes 248 feet in 8 seconds, and its speed at the end of 8 seconds is 55. Find the initial speed and the acceleration.
7. Draw a figure showing the distance traversed by a body in 10 seconds which starts with a speed of 30 feet per second in a certain direction, and is moving after 10 seconds with a speed of 20 feet per second in the opposite direction, the retardation having been uniform throughout the interval.

IV. *Composition and Resolution of Motions*

§ 25. We have hitherto considered the motion of a body moving with a uniform velocity or a uniform velocity increased or decreased by a uniform acceleration. We have now to consider cases of the motion of a body when more than one velocity is given to it, in the same straight line or in different directions. In these cases the motion is said to be compounded, and when the velocities are not in the same direction the actual motion is a compromise between them.

§ 26. **Resultant Velocity.**—If a body tend to move with several different velocities, the velocity with which it actually moves is called the *resultant* velocity, and those several velocities are called *components.* The process of finding the resultant velocity, when the component velocities are given, is called the *Composition of the Velocities,* whilst the converse process of finding component velocities which are equivalent to a given velocity is called *Resolution.*

Cases of the composition of velocities occur when a body is moving on something which is itself in motion, as when a boat is ascending or descending a stream, when a man is walking on the deck of a steamer, or when a stone is dropped from a moving balloon.

§ 27. **Composition of Uniform Velocities in the same Straight Line.**—If a body tend to move with a velocity u which would take it from A to B (fig. 7) in one second, and likewise with a velocity u' which would take it from B to C in the same straight line in one second, then at the end of the second the body will be found at C, as if it had moved with a velocity $u \pm u'$. So, too, if the body have several tendencies to uniform motion in the same straight line, the resultant velocity

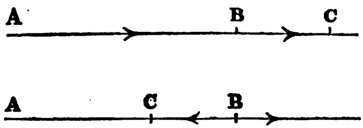

FIG. 7

will be the algebraical sum of the component velocities.

Suppose the velocity of a stream to be 3 miles an hour, and a vessel to be sailing at the rate of 8 miles an hour in still water, then the actual velocity of the vessel is 5 or 11 miles an hour, according as the vessel is sailing up or down stream. When a man paces up or down the deck of a steamer, which is sailing along a river, the actual velocity of the man is the algebraical sum of the velocity of the steamer and of the stream, and of the rate at which the man is walking.

§ 28. **Composition of Uniform and Accelerated Motions in the same Straight Line.**—In §§ 13 and 14 we have already considered the case of a uniform initial velocity combined with a uniform acceleration in the same direction, and we have seen that the velocity after t seconds is the initial velocity increased by the gain of velocity due to the acceleration, and further that the distance traversed is equal to the distance due to the initial velocity increased by that due to the acceleration. Now, as acceleration is only the velocity gained per second, acceleration in the same straight line may be compounded according to the same law as uniform velocities, and the resultant acceleration is equal to the algebraic sum of the component accelerations.

If, then, a body is moving with several uniform velocities, $u_1, u_2, u_3 \ldots$ and with several accelerations, $a_1, a_2, a_3, \ldots$ all in the same straight line, but not necessarily in the same direction, and if $U = u_1 + u_2 + u_3 + \ldots$

and $A = a_1 + a_2 + a_3 + \ldots$

then the conditions of motion are determined by substituting U and A for u and a in the fundamental formulæ

$$v - u = \pm a t$$
$$s = u t \pm \tfrac{1}{2} a t^2$$
$$v^2 - u^2 = \pm 2 a s.$$

§ 29. **Composition of Velocities not in the same Straight Line.**—If a body have two different velocities in different directions, at the same time, it is a matter of common experience that the actual motion is along neither of these directions, but along a line intermediate between them. Thus, if a man row a boat at right angles to the current of a river, the actual course of the boat is a line which crosses the river in a slant direction from one bank to the other.

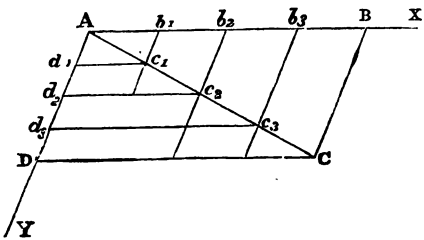

FIG. 8

Suppose, now, a body tend to move with a uniform velocity u which would take it from A to B (fig. 8) in one second, and with a uniform velocity u' which would take it from A to D in one second, then at the end of the second the body will be found at C, where BC is equal and parallel to AD. Moreover, the body will have moved along AC, and AC represents the resultant velocity. That AC will be the path of the body may be seen by supposing the body to be moving along AX whilst

the line $A X$ moves parallel to itself with its extremity in $A Y$. Then if $A B$ be divided into any number of equal parts, say four, and $A D$ into a like number of parts, while the body moves from A to b_1, the point A with the line $A B$ will move from A to d_1, and the body will be at c_1 at the end of the first quarter of a second; and for the same reason the body will be at c_2, c_3 at the end of each subsequent quarter of a second. The points c_1, c_2, c_3 can be proved to be in the same straight line by equality of triangles, and since $A c_1$, $c_1 c_2$, $c_2 c_3$ are equal, the motion along $A C$ is uniform.

§ 30. **Parallelogram of Velocities.**—The foregoing proposition is known as the *parallelogram of velocities*, and may be enunciated thus :—*If a body tend to move with two uniform velocities represented by the two sides of a parallelogram, drawn through a fixed point, then the resultant velocity will be represented by the diagonal of this parallelogram that passes through the same point.* This is sometimes called the *Parallelogram Law*, and will be found to be applicable to other quantities besides velocities.

§ 31. **Triangle Law.**—Now, if a body tend to move with two velocities represented by $A C$ and $C A$, it will remain at rest, since $A C = - C A$; and their algebraic sum, therefore, equals *zero*.

Since, then, the velocities $A\ B$ and $A\ D$ (fig. 8) are equivalent to $A\ C$, it is clear that if a body tend to move with three velocities represented by $A\ B$, $A\ D$, and $C\ A$, the body will remain at rest; and since $B\ C$ is equal to $A\ D$, the three velocities that neutralise one another can be represented by $A\ B$, $B\ C$, and $C\ A$—the three sides of a triangle taken in order.

Hence: *If a body tends to move with three velocities, that can be represented by the three sides of a triangle taken in order, the body remains at rest.*

§ 32. **Polygon Law.**—It follows from the foregoing that if a body tend to move simul-

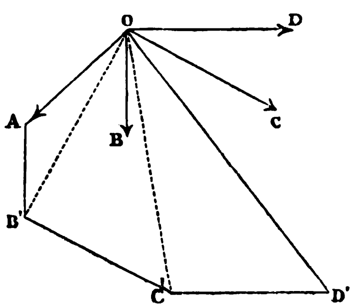

FIG. 9

taneously with several velocities, which would take it (fig. 9) from O to A, from O to B, from O to C, from O to D in one second, and if $A\ B'$ be drawn equal and parallel to $O\ B$, $B'\ C'$ equal and parallel

to $O\,C$, and $C'\,D'$ equal and parallel to $O\,D$, then, since $O\,B'$ is the diagonal of the parallelogram formed by $O\,A$, $O\,B$, it represents the resultant of these two velocities, and $O\,C'$ represents, for the same reason, the resultant of the velocities $O\,B'$ and $O\,C$, i.e. of $O\,A$, $O\,B$, and $O\,C$; and similarly $O\,D'$ represents the final resultant of the several velocities. We see, therefore, that if a body have these several tendencies to motion, it will be found at the end of a second or of any given time at the same point D', as if it had moved first from O to A, then from A to B', thence from B' to C', and finally from C' to D', i.e. along the sides of a polygon which respectively represent the velocities. And if the point D' had coincided with O, or the body had had an additional velocity represented in magnitude and direction by $D'\,O$, the body at the end of the second, or of any less period of time, would have been at O; in other words, it would have remained at rest. Hence:

If the several velocities, with which the body tends to move, can be represented in magnitude and direction by the sides of a closed polygon taken in order, the body will be at rest; but if the velocities are represented by the sides of an open polygon, the body will move, and the resultant velocity will be represented by the straight line that closes the polygon.

§ 33. **Composition of Accelerations not in the same Straight Line.**—If a body have two different accelerations along two straight lines, the distances $A b_1$, $b_1 b_2$ along $A B$ (fig. 8), and the distances $A\ d_1$, $d_1\ d_2$ along $A\ D$, traversed in equal times, will *not* be equal, but the distances $A\ d_1$, $A\ d_2$, &c., will be proportional to the distances $A\ b_1$, $A\ b_2$, &c., and, therefore, the points c_1, c_2, &c., will lie on a straight line, and the diagonal $A\ C$ will represent the resultant acceleration and also the path of the body. The *Triangle Law* and the *Polygon Law* hold good, therefore, in the case of accelerations, as in the case of uniform velocities.

§ 34. **Composition of Uniform Velocity and Acceleration.**—Suppose a body tend to move with

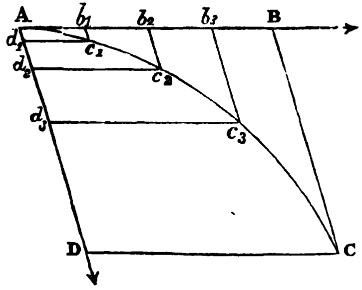

FIG. 10

a uniform velocity which would take it from A to B in one second, and likewise with an acceleration that would take it from A to D in

one second ; then at the end of the second the body will be found at C, where $B\,C$ is equal and parallel to $A\,D$, just as if it had moved from A to B and from B to C in the second ; but the body will *not* have moved along the diagonal $A\,C$. For, since the velocity along $A\,D$ is not uniform, the distances traversed in equal intervals of time along $A\,D$ will *not* be equal, whilst they are equal along $A\,B$, and the distances $A\,b_1$, $A\,b_2$, &c., will *not* be proportional to the distances $A\,d_1$, $A\,d_2$, &c., and therefore the points c_1, c_2, c_3 will *not* lie in a straight line. In this case, therefore, the path is a curve, and the nature of the curve depends on the magnitude of the acceleration. The path of a shot projected at a certain angle to the horizon is a curve resulting from the composition of a uniform velocity in one direction and a uniform acceleration in a different direction.

These results may be easily verified on a piece of squared paper by giving to u and a different values in the formulæ $s = u\,t$ and $s = \frac{1}{2}\,a\,t^2$. The distances $A\,b_1$, $A\,b_2$, &c., and $A\,d_1$, $A\,d_2$, &c., may thus be found, and by joining the corresponding points $A\,c_1$, $c_1\,c_2$, &c., the path of the body may be marked out. The results may be summarised as follows :—*When a body has several different velocities in different directions, the body will be, at the end of any given time, at the same point as if it had moved with each*

velocity separately. This is the fundamental law of the composition of motions, and it shows that all problems which involve simultaneous tendencies to motion may be treated as if those tendencies were successive.

§ 35. **Scalars, Vectors.**—The foregoing propositions are only special instances of a general law applicable to the composition of any quantities which have both magnitude and direction, and can consequently be represented by straight lines. Most physical quantities belong to one of two classes, according as they have or have not *direction.* Such quantities as *time, speed, volume,* and *density* have magnitude only, and these are called *scalars.* They can be represented by units of length, or units of area, or by other units without reference to *sign. Velocity,* and *acceleration,* and *force,* too, as we shall see later on, are not only measurable quantities, but have also a definite direction, which must be given, in order that they may properly be indicated. Such quantities are called *Vectors.* A velocity consists of a given number of units in a given direction; it may be *positive* or *negative*; but the *negative* sign, as indicating direction, cannot be applied to a volume.

§ 36. **Resolution of Velocity.**—As the diagonal of the parallelogram, the sides of which represent the component velocities, was found to represent

the resultant velocity, so any velocity represented by a certain straight line may be resolved into component velocities represented by the sides of the parallelogram of which that line is the diagonal. Suppose a bcdy start with a velocity that would take it from O to A in one second, then, if $OCAB$ be any parallelogram described on OA as diagonal, the body would equally be at A at the end of one second, if it had started with two velocities simultaneously, which would separately take it from O to C and O to B in one second, and therefore OB, OC represent the components of this velocity.

As the sides OC, OB may be inclined to each other at any angle, a given velocity may be resolved into two components in an infinite number of ways.

§ 37. **Rectangular Components.**—The most important case in practice is that in which the two components are at right angles.

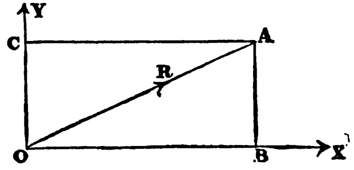

FIG. 11

If OB and OC be at right angles to each other, then $OA^2 = OB^2 + OC^2$, and if X be the

component along $O\,B$, and Y the component along $O\,C$, and if R be the original velocity along $O\,A$, we have

$$X : R :: O\,B : O\,A \text{ or } X = \frac{O\,B}{O\,A} R.$$

$$Y : R :: O\,C : O\,A \text{ or } Y = \frac{O\,C}{O\,A} R.$$

X and Y are called the resolved parts of R along $O\,X$ and $O\,Y$ respectively. If $O\,A$ represent any acceleration, then $O\,B$ and $O\,C$ will equally represent the *resolved parts* of this acceleration along $O\,X$ and $O\,Y$.

§ 38. **Examples.**—(1) A body tends to move with velocities of 30 feet and 40 feet per second along two straight lines at right angles to each other; find the resultant velocity.

Let V = resultant velocity,

then $V^2 = 30^2 + 40^2 = 2500 \therefore V = 50$ feet per second.

(2) A body is moving with an acceleration a; find the resolved parts of the acceleration along lines inclined to the direction of the acceleration at angles of 30°, 45°, 60° respectively.

If the angle $A\,O\,B$ is 45° it follows that $O\,B = B\,A$ and $O\,A^2 = 2\,O\,B^2$;

$\therefore O\,B = \frac{O\,A}{\sqrt{2}}$, and i a be the acceleration along $O\,A$, its component along $O\,B$ is $\frac{a}{\sqrt{2}}$.

If the angle $A\,O\,B$ is 30°, it follows that $O\,A = 2\,A\,B$,

$\therefore O\,A^2 = O\,B^2 + \frac{O\,A^2}{4}$, or $O\,B = O\,A\,\frac{\sqrt{3}}{2}$, and the component of the acceleration is $\frac{a\sqrt{3}}{2}$.

If the angle $A\,O\,B$ is 60°, $O\,B = \frac{A\,O}{2}$ and the component required is $\frac{a}{2}$.

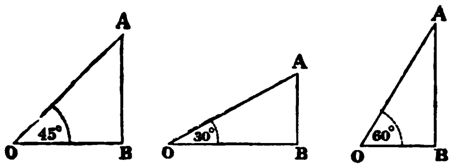

FIG. 12

If, instead of an acceleration, $O\,A$ represent a velocity V, the resolved parts of V along $O\,B$ will be, similarly, $\frac{V}{\sqrt{2}}$, $\frac{V\sqrt{3}}{2}$, and $\frac{V}{2}$ respectively in each of the preceding cases.

As these results frequently occur, they should be very carefully remembered.

EXERCISES IV

1. A fly is walking with a uniform velocity along a straight rod, which is itself being moved uniformly in a direction at right angles to its length. Determine the character of the fly's absolute motion.
2. The wind blows from a point intermediate between north and east. The northerly component of its

velocity is 10 miles an hour, and the easterly component is 36 miles an hour. Find the whole velocity.

3. A body is simultaneously urged to move with velocities of 50, 21, and 25 respectively; can the body remain at rest?
4. A body whilst moving vertically downwards with a uniform velocity of 10 feet per second is urged horizontally with an acceleration of 5; find its distance from starting-point after 2 seconds.
5. A body tends to move in a certain direction with an acceleration of 32, but is constrained to move in a direction inclined at an angle of 45° to the original direction; find the component of its acceleration in the latter direction.
6. A body moving with a uniform velocity of 30 miles an hour has its velocity accelerated each second 10 feet per second in the same direction; find the distance traversed in a quarter of a minute.
7. A body is moving at the rate of 40 miles an hour when its velocity is retarded each second at the rate of 6 inches per second; when and where will it stop?
8. A body tends to move with equal velocities of 10 feet per second in two directions inclined at 120° to each other; find its path and resultant velocity.
9. Two bodies start from *A* to *B* and from *B* to *A*, two points 80 yards apart, at the same time; the one moves uniformly at the rate of 10 feet per second, the other at the rate of 12 feet per second; where will they meet?
10. If a particle 10 inches from a given point revolve round it 7 times in 22 seconds, find the velocity of the particle.
11. Two men *A* and *B* start at the same moment in the same direction, from two points 1,500 feet apart; if *A* walk 4 miles an hour and *B* $3\frac{1}{2}$ miles an hour, where will *A* overtake *B*?
12. A train, having moved from rest, has acquired a velocity of 30 miles an hour in 5 minutes. Express

the acceleration, taking a foot and a second as the units of length and time.

13. A body begins to move with a velocity of 100 feet per second, and at the end of 7 seconds its velocity is 65. By how much is the velocity retarded each second?

14. Show how it is that the distance described in any time, when a body moves with a uniform acceleration, is proportional to the square of the time.

15. A body is simultaneously impressed with three uniform velocities, one of which would cause it to move 10 feet north in 2 seconds, another 12 feet in 1 second in the same direction, and a third 21 feet south in 3 seconds. Where will the body be in 5 seconds?

16. A body tends to move horizontally with a uniform velocity of 12 feet per second, and also vertically downwards with a uniform velocity of 8 feet per second; determine the position of the body after 3 seconds.

17. A body begins to move with an acceleration of 8 (feet, seconds), and its velocity is at the same time retarded 8 inches per second each second; find the distance traversed in 3 seconds.

18. Explain why it is dangerous to jump out of a railway carriage in motion.

19. A body is projected horizontally from the top of a vertical cliff with a velocity of 500 feet per second; it reaches the ground in 3 seconds; find its distance from the foot of the cliff.

20. If a person is walking in a straight line, in what direction must he throw a ball upwards, that it may return into his hand?

21. If a ball be thrown out of the window of a railway carriage in motion, in what direction will it seem to fall, and in what direction will it really fall?

22. A body starting from rest moves uniformly with a velocity of 10 feet per second, and also with an acceleration of 32 in the same direction; what

distance will be traversed in the 3rd second of its motion?

23. A body moves with a velocity of 10 feet per second in a given direction; find the velocity in a direction inclined at an angle of 30° to the original direction.

24. What acceleration along a certain line is equivalent to an acceleration of 20 in a direction that makes an angle of 45° with that line?

EXAMINATION QUESTIONS I

1. A balloon is carried along by a current of air moving from east to west at the rate of 60 miles an hour, having no motion of its own through the air, and a feather is dropped from the balloon. What sort of path will it appear to describe as seen by a man in the balloon?—*Univ. of Lond. Matric.*, June 1874.

2. A river 1 mile broad is running downwards at the rate of 4 miles an hour, and a steamer moving at the rate of 8 miles an hour wishes to go straight across. How long will the steamer take to perform the journey, and in what direction must she be steered?—*Ib.* Jan. 1875.

X

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3. At the earth's equator the hot air ascends, and is replaced by cold air which blows in along the ground from the poles. That which comes from our hemisphere blows from the north-east instead of from the north. Explain this.—*Ib.* June 1875.

4. Two bodies start together from rest, and move in directions at right angles to each other. One moves uniformly with a velocity of 3 feet per second; the other moves under the action of a constant force. Determine the acceleration due to this force if the bodies at the end of 4 seconds are 20 feet apart.—*Ib.* June 1878.

5. The space passed over in any time may be represented by an area. Explain clearly the meaning of this

statement, and under what conditions it is true. Show how to employ it to determine the space passed over by a body in 10 seconds after it starts from rest and has its velocity increased by 1 foot per second at the beginning of each second.—*Ib.* June 1878.

6. Explain a convenient method of representing geometrically the velocity of a body moving according to a known law, and the distance passed over by it. Employ the method to find the distance traversed in 10 minutes by a train which has a velocity of 20 miles an hour, and which has its speed diminished at a uniform rate of 5 miles an hour.—*Ib.* June 1882.

7. A ship is sailing due north at the rate of 4 feet per second; a current is carrying it due east at the rate of 3 feet per second; and a sailor is climbing a vertical mast at the rate of 2 feet per second. What is the velocity of the ship, and what the velocity of the sailor, relative to the sea-bottom?—*Ib.* Jan. 1883.

8. A train which is uniformly accelerated starts from rest, and at the end of 3 seconds has a velocity with which it would travel through 1 mile in the next 5 minutes. Find the acceleration.—*Ib.* June 1883.

9. A boat is rowed on the river so that its speed in still water would be 6 miles an hour. If the river flows at the rate of 4 miles an hour, show, by drawing a figure, how to find the direction in which the head of the boat must be kept in order that its motion may be at right angles to the current.—*Ib.* June 1885.

10. A body starts from rest and moves with uniform acceleration 18 (feet, seconds); find the time required by it to traverse the *first*, *second*, and *third* foot respectively.—*Ib.* June 1885.

11. What is meant by the statement that the acceleration of a particle is 32 foot-second units? With this acceleration how far will a particle move in 10 seconds, and what will be its velocity at the end of that time?—*Ib.* Jan. 1886.

12. A ship is sailing north-east with a velocity of 10 miles an hour, and to a passenger on board the wind appears to be blowing from the north with a velocity of $10\sqrt{2}$ miles an hour. Find the true velocity of the wind.—*Ib.* June 1888.

13. Explain how to compound velocities.

A person on an express train moving 60 miles an hour wishes to hit a stationary object which is situated 100 yards off in a line through the marksman at right angles to the line of motion of the train. If his bullet moves 1,200 feet per second, find out how much to one side of the object he should aim.—*Ib.* Jan. 1889.

14. If a point has a velocity of 1 foot per second to the east, and also a velocity of $\sqrt{3}$ feet per second to the north, determine the velocity which must be compounded with these to bring the point to rest.—*Ib.* June 1890.

15. A ship is sailing north at the rate of 8 miles an hour through the sea, and a man walks at the rate of 7 feet per second straight across her level deck on a line drawn at right angles to her length; draw a diagram (as well as you can to scale) by measuring which one might find the angle the man's resultant path makes with the north, and calculate his velocity with respect to the sea.—*Ib.* Jan. 1891.

16. The velocity of a body is increased uniformly in each second by 20 feet per second; by how many yards a minute will the velocity be increased in one minute? —*S. & A. Dep.* 1887.

17. When a particle is moving at the rate of 45 miles an hour, what would be the velocity if estimated in feet and seconds? Suppose the velocity to be acquired uniformly in 11 seconds, by how much is the velocity increased per second?—*Ib.* 1890.

18. If the acceleration of a body's velocity is 55 in feet and seconds, what is it in yards and minutes?—*Ib.* 1888.

19. Two bodies, whose velocities are accelerated in every second by 3 and 5 feet per second respectively, begin to move towards each other at the same instant and without having any initial velocity: at first they are 1 mile apart: after how many seconds will they meet?—*Ib.* 1889.

20. A particle whose velocity undergoes a constant acceleration starts from rest, and after describing 50 feet has a velocity of 20 feet per second; find the increase of its velocity per second, and the time in which it describes 50 feet.—*Ib.* 1890.

CHAPTER II

FALLING BODIES

V. *Bodies Falling Freely*

§ 39. The complete investigation of the motion of falling bodies is a branch of Kinetics and not of Kinematics. But everyday experience brings under our notice cases of bodies falling to the ground, and we can conveniently consider these cases as illustrating the principles established in the previous lessons, without reference to the force operating between the falling body and the earth, which is the cause of the motion. It will be shown later on that all bodies at the surface of the earth tend to move vertically downwards, with an acceleration of about 32·2 (feet, seconds), and that this acceleration is independent of the size of the body and of the quantity of matter it contains.

Common experience would lead us to suppose that a small ball of lead would fall more quickly than a similar ball of cork, because we are accus-

tomed to see light bodies, such as feathers, fall very slowly to the ground. A little thought, however, will show us that the resistance of the air must have more effect on large and light bodies than on small and heavy bodies, and it may easily be proved by trial that a feather and a ball of lead will fall to the ground in the same time in a vessel from which the air has been removed. It can be shown, however, that the acceleration varies with the distance of the body from the centre of the earth. Thus at the summit of a high mountain it is less than near the surface of the earth, and at the equator, in consequence of the peculiar configuration of the earth, it is less than in the neighbourhoood of the poles. As the velocity gained by a body falling freely is found to vary with the latitude of the place, and likewise with its height above the sea-level, and as, moreover, it differs numerically with the unit of length adopted, it is usually represented in books on Mechanics by the letter g. Neglecting the resistance of the air, we are able to state that all bodies acquire each second a velocity of g feet per second in falling to the ground, and that g varies with the distance of the body from the earth's centre, but is the same for all kinds of bodies. As the substance of the body does not need to be taken into consideration, all problems concerning falling bodies may be regarded as cases of accelerated motion in which

$a = g$, and may be solved by the application of the formulæ already established :

$$v - u = g\,t$$

$$\frac{v + u}{2} t = s ; \text{ or}$$

$$s = u\,t + \tfrac{1}{2} g\,t^2$$

$$v^2 - u^2 = 2\,g\,s,$$

where u is the initial velocity with which a body is projected *downwards.*

If the body is projected vertically *upwards,* u and g have opposite signs, and the body will evidently lose each second of its motion the same velocity which it would gain if it *fell* freely for one second. Hence, a body projected upwards may be said to be moving with a negative acceleration or retardation equivalent to a *loss* of velocity each second of g feet per second. The consequences which are involved in this proposition we will now proceed to consider separately.

§ 40. **To find the time during which a body rises when projected vertically upwards with a certain velocity.**

Let u be the velocity of projection, then

$$v = u - g\,t,$$

where v is the actual velocity of the body upwards at the time t. Now at the moment when the body reaches its highest point, its velocity equals zero. If, therefore, we put $v = 0$ in the above equation,

the corresponding value of t will be the time of rising;

$$\therefore t = \frac{u}{g}.$$

This shows that a body takes the same time to lose a velocity u in rising as to acquire it in falling freely from rest.

§ 41. **To find the whole time of flight.**

The whole time of flight is the time during which the body is in motion, i.e., in the case supposed, the time between the starting of the body vertically upwards with the velocity u, and its return to the point from which it was projected.

The formula $s = u\,t - \frac{1}{2}\,g\,t^2$ gives the distance of a body from the starting-point after t seconds, when projected vertically upwards with the velocity u. Now it is evident that when a body has risen to its maximum height and returned to the point of projection, its distance from the starting-point is zero, or $s = 0$. If, therefore, we put $s = 0$ in the above equation, we get t equal to the whole time of flight;

$$\therefore u\,t - \frac{g\,t^2}{2} = 0,$$

which gives $t = 0$ or $t = \frac{2\,u}{g}$. The former of these two values shows that $t = 0$ before the body starts, the latter that $t = \frac{2\,u}{g}$ when the body has

returned. Hence $\frac{2u}{g}$ is the *whole time* of flight. But $\frac{u}{g}$ has just been proved to be the time of rising, therefore $\frac{u}{g}$ must also be the time of falling; i.e. *the time of rising equals the time of falling.*

If v equal the velocity with which the body passes any point in its path when rising, then, since v at that point may be considered as a velocity of projection, the body will have the same velocity when it returns to that point. In other words, a body passes each point in its path with the same velocity, whether rising or falling. This proposition may be proved directly from the formula

$$v^2 = u^2 - 2gs,$$

where $v =$ velocity with which the body passes a point the distance of which from the point of projection is s. For, since

$$v^2 = u^2 - 2gs, \; v = \pm\sqrt{(u^2 - 2gs)}.$$

Hence v has two equal values differing in sign only, which shows that if $+v$ be the velocity with which the body passes any point in its path when rising, $-v$ will be the velocity with which it passes the same point when falling.

§ 42. **To find the height to which a body will rise when projected vertically upwards with a**

given velocity.—Take the formula $v^2 = u^2 - 2\,g\,s$; then, since $v = 0$ at the summit, the corresponding value of s equals the height to which the body will rise,

$$\therefore \quad u^2 = 2\,g\,s;$$

or

$$s = \frac{u^2}{2\,g}.$$

Since $u^2 = 2\,g\,s$, where s is the height to which a body rises, and u is the velocity of projection, we see that a body would rise through the same height in losing a velocity u, as it would fall through to gain it.

The student should exercise himself in employing the graphic method of proof which has been explained in Lesson III. to establish the foregoing propositions.

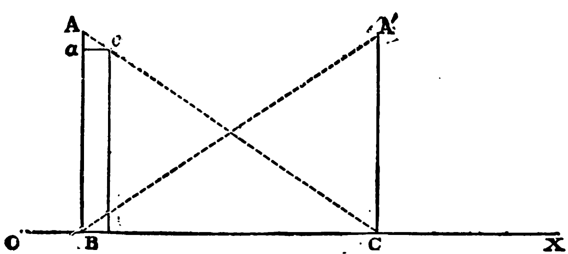

FIG. 13

Take the above proposition. If $A\,B$ represent u, the velocity of projection, $O\,X$ the line of time, and if $A\,a = g$, the acceleration, and

$a\,c$ represent one second, then if $A\,c$ be produced to C, $B\,C = t$, and the area $A\,B\,C$ represents the distance traversed. Now, if we make $C\,A' = B\,A$, $C\,A'$ represents the velocity acquired with acceleration g in time t, and, as the triangle $A\,B\,C$ is equal to the triangle $A'\,B\,C$, it is clear that the distance traversed in losing a velocity u, with a negative acceleration g, is equal to the distance traversed in gaining a velocity u, with a positive acceleration of equal value.

§ 43. **Values of g (Gravity Acceleration).**—It has been stated that the value of g varies with the latitude of the place, increasing as we pass from the equator to the poles. Expressed in feet and seconds it is approximately 32·2, and expressed in centimetres and seconds it is approximately 981 in the latitude of Greenwich. The following table gives more nearly the values of g at places situated at different latitudes :—

Place	Value of g	
	Feet, Seconds	Cents., Seconds
Equator	32·090	978·10
Paris	32·183	980·94
Greenwich	32·191	981·17
Berlin	32·194	981·25
Manchester	32·197	981·34
Edinburgh	32·203	981·54
Pole	32·254	983·11

§ 44. **Examples.**—(1) A body projected vertically downwards with a velocity of 20 feet per second from the top of a tower reaches the ground in 2·5 seconds; find the height of the tower. Take $g = 32{\cdot}2$. Then the distance traversed due to the velocity of projection is $20 \times 2{\cdot}5 = 50$ feet. The distance due to the gravity acceleration is $\frac{1}{2} g t^2 = 16{\cdot}1 \times (2{\cdot}5)^2 = 100{\cdot}625$ feet. Therefore the whole distance is 150·625 feet = the height of the tower.

(2) A body is projected vertically upwards with a velocity of 200 feet per second; find the velocity with which it will pass a point 100 feet above the point of projection.

Take $g = 32$.

$$\text{Here } u = 200,\ s = 100, \text{ and } v^2 = u^2 - 2gs$$
$$\therefore \quad v^2 = (200)^2 - 64 \times 100$$
$$= 40000 - 6400 = 33600$$
$$\therefore \quad v = \pm 40\sqrt{21}.$$

(3) A man is rising in a balloon with a uniform velocity of 20 feet a second, when he drops a stone which reaches the ground in 4 seconds; find the height of the balloon.

$$\text{Here } u = -20, \text{ and } t = 4$$
$$\text{and } s = -ut + \frac{gt^2}{2}.$$

Take $g = 32$;

$$\therefore s = -80 + 16 \times 16 = 176$$

$\therefore$ the height of the balloon was 176 feet.

Let us examine more in detail what happens in this case. The stone on leaving the balloon has an upward velocity of 20 and a downward acceleration of

32. It consequently rises for $\frac{5}{8}$ second, and has then reached its highest point. This point is $\frac{400}{64} = \frac{25}{4}$ feet above the point where it left the balloon. From its highest point it falls freely for $4 - \frac{5}{8} = 3\frac{3}{8}$ seconds, and during this time it falls through a distance equal to half the acceleration multiplied by the square of the time $= 16\left(\frac{27}{8}\right)^2$ feet. Hence the height of the balloon when the man drops the stone is the distance through which the stone falls from its highest point *less* the height to which it rises $= 16\left(\frac{27}{8}\right)^2 - 16\left(\frac{5}{8}\right)^2 = 176$ feet.

The student will derive advantage by solving problems by a method similar to this instead of by direct application of the formula.

(4) A man, standing on a platform which descends with a uniform velocity of 40 centimetres per second, drops a stone which reaches the ground in 5 seconds. From what height did he drop the stone?

Take $g = 980$ (centimetres, seconds).

Here $u = 40$, $t = 5$, and $s =$ the height

$$\therefore s = ut + \tfrac{1}{2} g t^2 = 5 \times 40 + 490 \times 25 = 200 + 12250$$
$$= 124 \cdot 5 \text{ metres.}$$

EXERCISES V

In the following exercises, unless otherwise stated, g may be taken as equal to 32 (feet, seconds) or 980 (centimetres, seconds).

1. Through what distance must a body fall to acquire a velocity of 80 feet per second?
2. A body falls freely for 5 seconds; what is its velocity?
3. A body falls freely for 6 seconds; through what distance will it fall in the last second and in the whole time?
4. A body is projected upwards with a velocity of 160 feet per second; to what height will it rise?

5. A body is projected upwards with a velocity of 80; after what time will it return to the hand?
6. A ball is thrown downwards with a velocity of 20 feet per second; find its distance from the point of projection after 3 seconds.
7. A ball is thrown downwards with a velocity of 50 feet per second; what is its velocity after 4 seconds?
8. With what velocity must a body be projected vertically upwards that it may rise 40 feet?
9. A body projected vertically upwards passes a certain point with a velocity of 80 feet per second; how much higher will it ascend?
10. A body projected horizontally from the top of a cliff with a velocity of 40 feet per second strikes the ground after 3 seconds; find the distance of the point of fall from the point of projection.
11. A man descending uniformly the shaft of a mine with a velocity of 100 feet per minute drops a stone which reaches the bottom in 2 seconds; through what distance did it fall?
12. A body starts with a velocity of 90 feet per second and loses each second 30 feet per second; how far will it move?
13. Two balls are dropped from the top of a tower, one of them 3 seconds before the other; how far will they be apart 5 seconds after the first was let fall?
14. If a body after having fallen for 3 seconds break a pane of glass, and thereby lose one-third of its velocity, find the entire distance through which it will have fallen in 4 seconds.
15. With what velocity must a body be projected vertically downwards that it may fall through 296 feet in 4 seconds?
16. A body falls freely; find the distances traversed in the 2nd, 5th, and 7th seconds respectively.
17. A body projected vertically downwards with a certain velocity traverses 120 feet in a certain second; find the distance traversed in the preceding second.

18. Two seconds after a body is let fall another body is projected vertically downwards with a velocity of 100 feet per second; when will it overtake the former?

19. With what velocity must a body be projected vertically upwards to return to the hand after 6 seconds?

20. A ball is dropped from the top of the mast of a ship which is sailing at the rate of 18 miles an hour; if the mast be 64 feet high, how far will the ship have sailed during the passage of the ball?

21. With what velocity must a ball be projected vertically upwards, to just reach the top of a tower 144 feet high, and how long will it take to reach it?

22. How high will a body rise projected upwards with a velocity of 96·6 feet per second? What will be its velocity $3\frac{1}{2}$ seconds after it was projected? [$g = 32 \cdot 2$.]

23. A man is standing on a platform which descends with a uniform acceleration of 5 (feet, seconds); after having descended for 2 seconds he drops a ball; what will be the velocity of the ball after 2 more seconds?

24. A balloon is rising uniformly with a velocity of ~~10~~ 20 feet per second, when a man drops from it a stone which reaches the ground in 3 seconds: find the ✓ height of the balloon, first, when the stone was dropped; and, secondly, when it reached the ground.

25. A stone falls freely for 3 seconds, when it passes through a sheet of glass, in consequence of which it loses half its velocity; find the height of the glass from the ground, if it reaches the earth 2 seconds after breaking the glass.

26. A body is projected vertically upwards from the top of an eminence with a velocity of 100 feet per second; find its velocity after 7·5 seconds.

27. With what velocity must a body be projected vertically upwards to attain a height of 40,401 feet?

28. A body under the action of gravity falls freely through 100·5 feet in 2·5 seconds; what is the value of g—

(1) when the units of time and space are *one second* and *one foot*, (2) when they are *one minute* and *one yard*?

29. A stone is projected downwards from the top of a precipice with a velocity of 25 metres per second; when will it have acquired a velocity of 74 metres per second, and what distance will it have then traversed? [$g = 9{\cdot}8$ (metres, seconds).]

VI. *Special Cases of Falling Bodies*

§ 45. **To determine the motion of a body projected from a given height with a uniform horizontal velocity.**—If a body is projected, say,

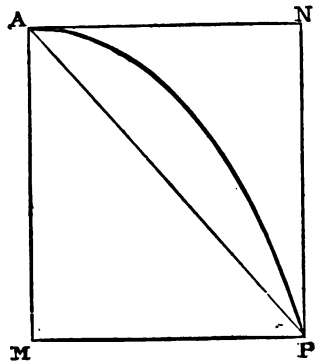

FIG. 14

from the top of a cliff with a given velocity in a horizontal direction, the velocity of projection and the gravity acceleration are at right angles to each

other, and the body will consequently move in a curve which may be plotted out on squared paper. The position of the body at any time may be found by ascertaining the distance of the corresponding point of the curve from the horizontal and vertical lines through A, the point of projection.

Let $u =$ the velocity of projection. Then if the body is at P after t seconds, $A N$ represents the distance the body would traverse in t seconds with the uniform velocity u, and $N P$ the distance it would move through in t seconds with acceleration g.

$\therefore$ $A N = u\, t$ and $N P = A M = \frac{1}{2} g\, t^2$. Hence the distance $A P = \frac{t}{2} \sqrt{4 u^2 + g^2 t^2}$.

§ 46. **To determine the motion of a body that starts with a uniform velocity in a direction inclined to that of the vertical at an angle less than a right angle.**—Let $A O$ (fig. 15) be the direction in which a body starts with a uniform velocity u. Then if points a_1, a_2, a_3 along $A O$ be so taken that $A a_1 = a_1 a_2 = a_2 a_3 = u$, and if verticals $a_1 b_1$, $a_2 b_2$, $a_3 b_3$ be drawn from a_1, a_2, a_3 equal in length to the distances through which a body falls freely in one second, two seconds, and three seconds respectively, the points b_1, b_2, b_3 determine the position of the body after each successive second, and in this way the path of the body may be plotted

out. At any point P which the body reaches after t seconds the distance $A\,Q = u\,t$, and the distance $Q\,P = \frac{1}{2}\,g\,t^2$, as before.

After a time, depending on the magnitude and direction of u, the body will be at a point B in the same horizontal line with A. If the body comes

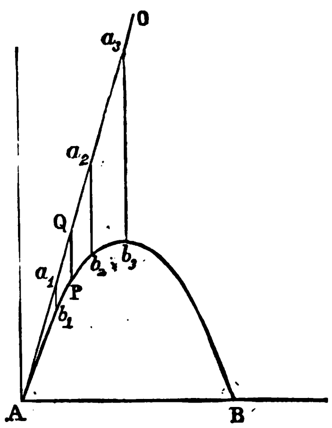

FIG. 15

to rest at B the time occupied in passing from A to B is called the whole time of flight, and the distance $A\,B$ is called the *horizontal range.*

§ 47. **To find the whole time of flight, the horizontal range, and the height to which the body rises.**—By applying the principle of the Resolution of Velocity (§ 37), the velocity u may

be resolved into a vertical and a horizontal component. Call the horizontal component X, and the vertical component Y. The horizontal component represents the distance through which the body moves horizontally in each second ; and the vertical component is that which is lessened by g each second as the body rises, and increased by g each second as the body falls.

If, then, t be the time the body takes to reach its highest point, $Y = t\,g$, and consequently (§ 41) the whole time of flight $= \dfrac{2\,Y}{g}$.

If h be the height to which the body rises before it begins to fall, $Y^2 = 2\,g\,h$, or $h = \dfrac{Y^2}{2g}$.

As $2\,t$ is the time occupied in passing from A to B, the distance $A\,B$ is $2\,X\,t$, where X is the horizontal component of the velocity; and as $t = \dfrac{Y}{g}$ the horizontal range $= \dfrac{2\,X\,Y}{g}$.

For the general determination of the horizontal and vertical components of the velocity a knowledge of trigonometry is necessary, but in some few cases they can be determined by ordinary geometric methods.

§ 48. **Examples.**—(1) A body on a level plane has simultaneously imparted to it a vertical velocity of 64

feet per second, and a horizontal velocity of 30 feet per second; find the horizontal range.

Here $X = 30$ and $Y = 64$, $\therefore 2t = \frac{128}{32} = 4$, taking $g = 32$, and horizontal range $= 2tX = 4 \times 30 = 120$ feet.

(2) A shot is fired in a direction inclined at 60° to the horizon with a velocity of 49 metres per second; find the height to which it rises and the horizontal range. By § 47 the vertical component of the velocity is $24{\cdot}5 \times \sqrt{3}$, and the horizontal component is 24·5 metres per second.

If h be the greatest height, $(24{\cdot}5 \times \sqrt{3})^2 = 2gh$;

$$\therefore h = \frac{24{\cdot}5 \times 24{\cdot}5 \times 3}{2 \times 9{\cdot}8}, \text{ taking } g = 9{\cdot}80 \text{ (metres, secs.)};$$

$$\therefore h = 91\tfrac{7}{8} \text{ metres.}$$

Similarly $24{\cdot}5 \times \sqrt{3} = tg$, where t is time of rising;

$$\therefore 2t = \frac{2 \times 24{\cdot}5 \times \sqrt{3}}{9{\cdot}8}$$

and the horizontal range $= \dfrac{2 \times (24{\cdot}5)^2 \times \sqrt{3}}{9{\cdot}8}$

$$= \frac{245\sqrt{3}}{2} \text{ metres.}$$

§ 49. **Motion on an Inclined Plane.**—In the preceding paragraphs we have considered cases of the combination of the vertical acceleration due to gravity with the whole of the resolved part of a uniform velocity of projection. We have now to consider cases of the motion of a body due to a resolved part of the vertical gravity acceleration, either alone or in combination with uniform velocity.

Examples of this kind of motion are seen when a body falls down or is projected along an inclined plane.

§ 50. Let $A B$ be a smooth plane inclined to the horizon at an angle $A B C$; it is required to find the *effectual acceleration* with which a heavy body will move if it slides down or is projected up the plane.

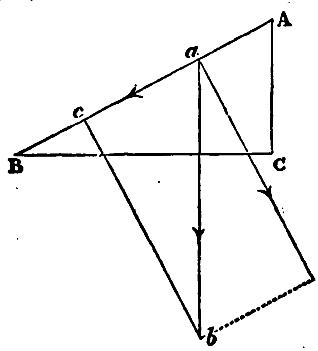

FIG. 16

Let the body be moving down the plane.

Suppose the body to be at a.

Draw $a b$ vertically downwards and make it equal to $A B$. Let $a b$ represent g, the vertical acceleration with which the body would move if free to fall. Then, if $b c$ be drawn at right angles to $A B$, the triangle $a b c$ is in every respect equal to the triangle $A B C$, and $a c$ represents the

resolved part of the acceleration $a\,b$ in the direction $A\,B$.

Let a be this component of g.

Then $a : g :: a\,c : a\,b$ (§ 37),

$$\text{or } a = \frac{a\,c}{a\,b} \cdot g = \frac{A\,C}{A\,B} \cdot g,$$

since $a\,c = A\,C$ and $a\,b = A\,B$.

Let $A\,C$, the height of the plane, equal h, and $A\,B$, the length of the plane, equal l.

$$\text{Then } a = \frac{h}{l} \cdot g.$$

If the angle of the plane is 30°, $a = \frac{g}{2}$

" " 45°, $a = \frac{g}{\sqrt{2}}$

" " 60°, $a = \frac{g\sqrt{3}}{2}$

If the body be projected up the plane the retardation or negative acceleration due to the body's tendency to fall will also be represented by $a\,c$, and will be equal to $-\frac{h}{l} \cdot g$.

§ 51. The motion of the body on a smooth inclined plane can be determined by substituting this value of a in the fundamental equations of § 16, the value of u being zero, if the body slide down the plane with the resolved part of the gravity acceleration only, and where the signs of

u and a are the same or opposite according as the initial velocity is downwards or upwards.

§ 52. **Examples.**—(1) Find the velocity with which a body must be projected up an inclined plane, the height of which is h feet, and length l feet, so as just to reach the top.

At the summit $v = 0$, $\therefore u^2 = 2as = 2\dfrac{h}{l}.g.l$;

$\therefore u = \sqrt{2hg}$, i.e. the same velocity as would be needed to project the body from C to A (fig. 16).

(2) Find the time occupied in falling down the whole length of an inclined plane.

Here $u = 0$, $s = l$, and $a = \dfrac{h}{l}g$;

$$\therefore l = 0 + \tfrac{1}{2}\frac{h}{l}gt^2;$$

$$\text{or } t = l\sqrt{\frac{2}{hg}}$$

showing that if the height of the plane remains the same, the time of falling varies directly with the length.

§ 53. To find the time of falling down any chord of a vertical circle drawn through its highest point.

Let AC (fig. 17) be the chord, AB the vertical diameter of the circle. Join CB and draw CD perpendicular to AB. Then the acceleration down $AC = \dfrac{AD}{AC}g = \dfrac{AC}{AB}g$, by similarity of triangles,

And,

As $u = 0$, and $s = A\,C$, and $a = \frac{A\,C}{A\,B} \cdot g$,

we have $A\,C = \frac{1}{2}\,a\,t^2 = \frac{1}{2}\,\frac{A\,C}{A\,B} \cdot g\,t^2$;

$$\therefore \frac{2\,A\,B}{g} = t^2 \therefore t = \sqrt{\left(\frac{2\,A\,B}{g}\right)}$$

which is constant, being independent of $A\,C$, and

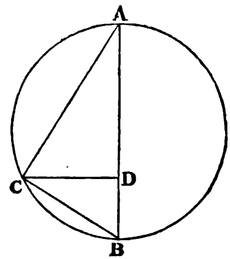

FIG. 17

shows that the time of falling down any chord is the same as the time of falling down the diameter.

§ 54. This proposition enables us to find the line of quickest descent from a point to a curve or from one line to another.

The line of quickest descent from a point to a straight line inclined to the horizon may be found thus :—Let P be the point (fig. 18), $A\,B$ the straight line. Through P draw $P\,A$, a horizontal line meeting $A\,B$ in A. Bisect the angle $P\,A\,B$ by $A\,O$.

Through P draw PC vertical, and from C draw CQ perpendicular to AB. PQ is the line of quickest descent. For it is evident that a circle may be described which has C for its centre and which touches AB and AP in Q and P; and since the time of falling down all chords of this circle from P is the same, PQ must be the line of quickest descent.

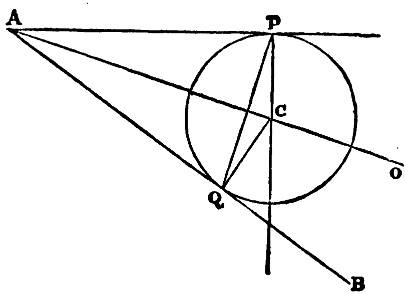

FIG. 18

The problem of finding the line of quickest descent from a point to a curve is thus found to resolve itself into the geometrical problem of drawing a circle, the highest point of which shall be the given point, and which shall touch the given curve.

EXERCISES VI

1. The angle of a plane is 30°, the length 20 feet; find the time occupied in falling from the top to the bottom.

2. The angle of a plane is 30°; find the velocity with which a body must be projected up it, to reach the top, the length of the plane being 20 feet.
3. Find the velocity of a body that has fallen for 3 seconds down a plane that rises 2 inches in a length of 5 inches.
4. A body is projected down a plane the inclination of which is 45° with a velocity of 10 feet; find the distance traversed in $2\frac{1}{2}$ seconds.
5. A steam-engine starts on a downward incline of 1 in 200 with a velocity of $7\frac{1}{2}$ miles an hour; neglecting friction, find the distance traversed in 2 minutes.
6. A body projected up an incline of 1 in 100 with a velocity of 15 miles an hour just reaches the summit; find the time occupied.
7. A body is projected with a velocity of 200 feet per second in a direction inclined to the horizon at 45°; find the greatest height it reaches, and the distance of the point where it touches the ground from the point of projection.
8. Find the distance traversed by a body that slides on a plane inclined at 30° to the horizon while the velocity changes from 48 to 16 feet per second.
9. A heavy body on a level plain has simultaneously communicated to it an upward vertical velocity of 48 feet per second, and a horizontal velocity of 25 feet per second. Find its greatest height, its range, and its whole time of flight.
10. A stone projected horizontally with a velocity of 20 feet per second from the top of a tower strikes the ground after 3 seconds; find the distance of the point of fall from the point of projection.
11. A body projected at an angle of 60° to the horizon with a velocity of 400 feet per second strikes the perpendicular face of a tower at a horizontal distance of 200 feet from the point of projection. Find the height at which it strikes the tower. ($g = 32$.)

EXAMINATION QUESTIONS II

1. A heavy particle is dropped from a given point, and after it has fallen for one second another particle is dropped from the same point. What is the distance between the two particles when the first has been moving during 5 seconds?
2. Through what vertical distance must a heavy body fall from rest in order to acquire a velocity of 161 feet per second? If it continue falling for another second after having acquired the above velocity, through what distance will it fall in that time? ($g = 32{\cdot}2$.)
3. A balloon has been ascending vertically at a uniform rate for 4·5 seconds, and a stone let fall from it reaches the ground in 7 seconds; find the velocity of the balloon and the height from which the stone is let fall.
4. If a heavy body is thrown vertically up to a given height, and then falls back to the earth, show that, neglecting the resistance of the air, it passes each point of its path with the same velocity when rising and when falling.—*Univ. Lond. Matric.*, Jan. 1871.
5. A ball is allowed to fall to the ground from a certain height, and at the same instant another ball is thrown upwards with just sufficient velocity to carry it to the height from which the first one falls; show when and where the two balls will pass each other.—*Ib.* Jan. 1871.
6. The intensity of gravity at the surface of the planet Jupiter being about 2·6 times as great as it is at the surface of the earth, find approximately the time which a heavy body would occupy in falling from a height of 167 feet to the surface of Jupiter.—*Ib.* Jan. 1873.
7. If a body is projected upwards with a velocity of 120 feet in a second, what is the greatest height to

which it will rise, and when will it be moving with a velocity of 40 feet per second?—*Ib.* Jan. 1874.

8. Suppose that at the equator a straight hollow tube were thrust vertically down towards the centre of the earth, and that a heavy body were dropped through the centre of such a tube. It would soon strike one side; find which, giving a reason for your reply.—*Ib.* June 1874.

9. From a point in a smooth inclined plane a ball is rolled up the plane with a velocity of 16·1 feet per second. How far will it roll before it comes to rest, the inclination of the plane to the horizon being 30°? Also, how far will the ball be from the starting-point after 5 seconds from the beginning of the motion? [$g = 32{\cdot}2$.]—*Ib.* June 1874.

10. A rifle-bullet is shot vertically downwards from a balloon at the rate of 400 feet per second. How many feet will it pass through in 2 seconds, and what will be its velocity at the end of that time, neglecting the resistance of the air and estimating the acceleration due to gravity as 32?—*Ib.* Jan. 1875.

11. A stone is let fall from the top of a railway-carriage which is travelling at the rate of 30 miles an hour. Find what horizontal distance and what vertical distance the stone will have passed through in one-tenth of a second.—*Ib.* June 1875.

12. A body projected vertically upwards against gravity has risen 120 feet in one second. What was its initial velocity of projection, and how far will it rise during the next second?—*Ib.* Jan. 1876.

13. A stone projected vertically upwards reached the ground again in six seconds. What was its height above the ground at the end of the first second? ($g = 32$.)—*Ib.* June 1876.

14. A stone is thrown into the air, at an angle of 45° to the horizon, with a velocity of 128 feet per second. Show that the path of the stone will not be a straight line; and determine the amount of vertical

deviation from a straight line at the end of two seconds, neglecting the resistance of the air. ($g=32$.) —*Ib.* June 1876.

15. A particle is projected in a horizontal direction with a velocity of 10 miles per hour, and at the same time falls under the action of gravity. Assuming that no other forces are acting, and taking $g=32$ (feet, seconds), draw a picture representing the position of the particle at the end of 1, $1\frac{1}{2}$, $2\frac{1}{4}$, and 3 seconds.—*Ib.* Jan. 1877.

16. What is the average velocity of a body during the first second of its fall under gravity; also during the first 2 seconds? Show how, by knowing the average velocity, you can find the whole space fallen through.—*Ib.* Jan. 1878.

17. A stone dropped into a well reaches the water with a velocity of 80 feet per second, and the sound of its striking the water is heard $2\frac{7}{12}$ seconds after it is let fall. Find from these data the velocity of sound in air. ($g=32$.)—*Ib.* Jan. 1879.

18. A body is projected up a smooth inclined plane, whose height is one half of its length, with a velocity of 60 feet per second, and just reaches the top. Find the length of the plane and the time taken in the ascent.—*Ib.* July 1879.

19. If $g=981$ centimetre second units, from what height must a body fall in order that it may have a velocity of 50 metres per second on striking the ground?—*Ib.* Jan. 1880.

20. A heavy body starting from rest slides down a smooth plane inclined 30° to the horizon. How many seconds will it occupy in sliding 240 feet down the plane, and what will be its velocity after traversing this distance? ($g=32$.)—*Ib.* June 1880.

21. The height of an inclined plane is $\frac{3}{5}$ of its length; a body is projected up the plane from the bottom with a velocity of 50 feet per second, and slides down again. Find the distance attained, and the

time before the body arrives at the starting-point.—*Ib.* June 1881.

22. Define acceleration. If a ball slides without friction down an inclined plane, and in the 5th second after starting passes over 2207·25 centimetres, find its acceleration and the inclination of the plane to the horizon. $g = 981$ (cms., secs.)—*Ib.* Jan. 1884.

23. A cannon-ball is shot horizontally from the top of a tower 49 feet high, with a velocity of 200 feet per second. Find at what distance from the tower the cannon-ball will strike the ground.—*Ib.* June 1885.

24. A stone is projected into the air with a velocity of 200 feet per second, in a direction inclined at 60° to the horizontal plane. With what velocity must another stone be projected vertically upwards that the two stones may rise to the same height above the horizontal plane?—*Ib.* Jan. 1886.

25. A stone is thrown vertically upwards with such a velocity as will just raise it to the top of a tower 100 feet high. Two seconds afterwards another stone is thrown up from the same place with the same velocity. Determine when and where the stones will meet.—*Ib.* Jan. 1887.

26. A stone is projected up a smooth inclined plane with a velocity which will just carry it to the top.. How far relatively would a stone ascend a plane of the same length but twice the height, if projected with half the velocity?—*Ib.* June 1887.

27. A body is projected vertically upwards from a point at a height h above the ground with a velocity due to a fall through a distance $\frac{1}{2}h$. Find when it will strike the ground.—*Ib.* Jan. 1888.

28. Find the ratio of the height to the length of a smooth inclined plane down which when a particle slides the acceleration of its velocity is $\frac{1}{5}$ of the acceleration of the velocity of a body falling freely under the action of gravity.—*S. & A. Dep.*, 1887.

29. If a body falling freely describes a distance of 30 yards in half a second, what was its velocity at the beginning of the half-second?—*Ib.* 1888.

30. A body slides down a smooth inclined plane, the height of which is 10 feet and the length 100 feet; find—

(*a*) The acceleration of the body's velocity while sliding;

(*b*) The velocity which the body acquires in sliding from the top to the bottom of the plane;

(*c*) The time it takes, starting without initial velocity, to go from the top to the bottom. ($g = 32$.)—*Ib.* 1890.

DYNAMICS—FORCE

KINETICS

CHAPTER III

MEASUREMENT OF FORCE

VII. *Explanation of Terms*

We have now to consider motion in its connection with the quantity of matter moved, and with the cause producing it.

§ 55. **Matter, Inertia.**—In treating, in the previous lessons, of motion in the abstract, we have had constantly to refer to moving bodies. In the whizzing bullet, in the running stream, and in the wind that blows we have examples of matter in motion. Whenever we speak of motion it must be in reference to moving matter. It is not easy to find a ready answer to the question, What is matter? Our perception of matter arises from the muscular feeling of resistance, and, consequently, whatever produces the feeling of resistance we call matter. The feeling of resistance implies force; hence the ideas answering to the terms matter and force are intimately associ-

ated. This connection is seen in the fact that it is easier to move an empty waggon than one filled with goods, and generally that the more matter we have to set in motion the more force we have to exert. As we ourselves are conscious of putting forth effort to change the motion of moving bodies, or to set in motion matter apparently at rest, so we speak of inanimate bodies which are capable of overcoming resistance as exercising force. In this way we speak of the force of a bullet in overcoming the resistance of a target, or the force of a stream and of the wind in overcoming the resistance of a wheel. This property of matter, universally present, by which it requires force to overcome its resistance to change of motion, is called *Inertia*, and serves as a measure of the quantity of matter in a body. Thus there is more inertia in a cannon-ball than in a grape-shot, in an ordinary cricket-ball than in a ball of the same size made of cork.

§ 56. **Mass and Density.**—The quantity of matter a body contains is called its *mass*, and, consequently, the *mass* of a body is measured by its *inertia*, i.e. by the force required to give it a definite velocity in a given time.

If we take two bodies of the same material—say, a cannon-ball and a grape-shot—the ratio of their masses, as measured by their motion, is the

ratio of their volumes. But if we take a ball of lead and a ball of wood of the same size, we find, by applying the same force to each, that the inertia, and consequently the mass, of the ball of lead is greater than the mass of the ball of wood. This difference in the structure of the substance of the two bodies is termed a difference in *density*, and is due to the difference in the closeness with which the particles are packed. The *mass* or quantity of matter in a body depends, therefore, not only on its size or *volume*, but also on its *density*; and the proper measure of density will be the quantity of matter in some definite volume of the body. The most convenient volume to take is the unit-volume, and therefore we define *density as the mass of a unit-volume*, always remembering that the mass must be ultimately measured by its inertia or resistance to motion or change of motion.

If, then, we call d the mass of a unit-volume, and M the mass of V units of volume, we have

$$M = Vd; \quad \text{or, } d = M \div V.$$

The unit of mass is an arbitrary quantity of matter, and is taken as the quantity of matter in a standard *pound avoirdupois*, or in a *gram*,[1] ac-

[1] The *gram* is the quantity of matter in a cubic centimetre of pure water at a temperature of 4° C. One pound avoirdupois equals 453·59 grams.

cording as we adopt the British or what is known as the *centimetre-gram-second* (C.G.S.) system of measurement.

Mass is invariably expressed in terms signifying weight, but the distinction between mass and weight must be carefully kept in view, and will be further considered later on, when we shall explain under what conditions this is correct. When we speak of a pound of lead, the word 'pound' expresses a definite quantity of matter. Commercially, weight always stands for mass, and the merchant who estimates his stock by cwts. and tons understands by those weights nothing more than the measure of the quantity of matter he possesses. When we use the word 'pound' to signify weight we shall call it a 'pound-weight.'

§ 57. **Momentum.**—Hitherto we have treated motion in the abstract, with reference to its rate and direction only, but without reference to the quantity of matter moved; and the rate of motion or velocity was in that case its correct measure. But it is evident that if two bodies are moving with the same velocity there will be a greater quantity of motion in that which contains the greater quantity of matter, just as there is more heat in ten gallons of water at 10° C. than in one gallon at the same temperature. When the motion of a definite amount of matter

is thus considered, it is called *momentum.* Thus, the word momentum is employed to express the *quantity of motion in a moving body.* The difference between momentum and velocity is analogous to that which exists between the quantity of heat a body contains and its temperature. Everyone knows that there is more heat in a hundred gallons of water at 20° C. than there is in a teaspoonful of boiling water, although the temperature of the latter is much higher than that of the former.

In measuring momentum it is necessary to take some fixed amount of motion as a unit. The *unit of momentum* is defined as *the quantity of motion in a unit of mass moving with a unit of velocity.* The unit of mass in this country is the quantity of matter in a standard pound avoirdupois or in a gram, and the unit of velocity has been already defined. According to the units we adopt, therefore, the unit of momentum is the quantity of motion in a pound moving with a velocity of one foot per second, or the quantity of motion in a gram moving with a velocity of one centimetre per second.

If a body contains M units of mass, and is moving with a unit of velocity, it will possess M units of momentum, and if it is moving with v units of velocity it will possess $M v$ units of mo-

mentum. This is what is meant by saying that the momentum of a body whose mass is M and velocity v equals $M v$.

§ 58. **Force.**—We are now in a position to consider what is meant by force. The principal properties of matter with which we are concerned are, that it moves, and has inertia, or offers resistance to change of motion. Now, force is the name given to the unknown causes of all the various phenomena which affect these properties; and as all these phenomena are accompanied by motion, or tendency to motion, we define *force as whatever produces, or tends to produce, motion or change of motion.* It is evident that of forces, *per se*, we can know absolutely nothing. We can only observe their effects; and of these the most general is motion. We have already seen that matter and the tendency to motion are always conjoined, and this fact has led some writers to identify force and matter. It is quite certain that matter does not exist apart from force; and we need not now pause to consider whether force can exist apart from matter.

§ 59. **Measurement of Force.**—The intimate connection that exists between moving matter and force enables us to measure force by the amount of motion produced or destroyed in a unit of time. Having already defined what is meant

by quantity of motion, we see that *the proper measure of force is the momentum generated or destroyed in one second.*

§ 60. **Unit of Force.**—The unit of force is that force which in a unit of time can produce or destroy a unit of momentum. In the British system *the unit of force is that force which, acting for one second, will give to a mass of one pound a velocity of one foot per second.* This unit is called a *poundal.* In the C.G.S. system *the unit of force is that force which, acting for one second, will give to a mass of one gram a velocity of one centimetre per second.* This unit is called a *dyne.* Both the *poundal* and *dyne* are absolute units of force.

§ 61. **Force of Gravitation.**—The forces of nature are very various, being due to gravitation, cohesion, heat, electricity, and other causes. All forces tend to produce motion, some acting between bodies widely distant in space, and others between the molecules of bodies in intimate juxtaposition. Of these forces that which is most easily measured is the force of gravitation. This force is universally present, and gives rise to the phenomenon known as weight. The force of gravitation acts between all bodies, and through any intervening space. The law of universal gravitation was established by Newton. It asserts that every particle of matter attracts,

and is attracted to, every other particle of matter, wherever situated; and that the attractive force diminishes as the square of the distance between the two bodies increases. The force of gravitation explains the motion of the planets and other heavenly bodies, as well as the tendency of all bodies near the earth to fall to the ground. This force, being universally present, we can use it in certain cases as a standard of comparison for other forces. It should, however, be clearly understood that *weight* is but one of many different forces, which, under certain conditions, it may serve to measure. Just as gold, which is no more wealth than any other commodity, is taken to represent and measure other kinds of wealth, on account of its easy divisibility, its constancy of value, and other properties, so weight is frequently taken as the standard by which other physical forces are estimated. A sovereign may be regarded as a certain amount of wealth in the form of gold, and also as the measure of an equal amount of wealth in some other form; and in the same way a pound-weight, which represents a definite amount of the force due to gravity, may serve also to measure other forces capable of producing the same effect.

§ 62. **Weight and Mass.**—We have seen that by the *weight* of a body is understood the amount

of gravitation force acting on the body, and by *mass* the quantity of matter the body contains. The word weight, however, has been made to do double duty, sometimes standing for force and sometimes for mass. These two significations of the same word should be carefully distinguished. Since the force of gravitation varies inversely with the square of the distance from the earth's centre, the same amount of matter is heavier at one place than at another. Hence, whilst the mass of a body remains the same, its weight may change. Thus the same amount of matter weighed in a spring balance would be found to stretch the spring less at the equator than at the poles, and less also at the top of a high mountain than at the level of the sea. Whenever weight is used to measure mass it is tacitly understood that the gravitation force is constant, and on this supposition only can weight become a correct measure of the quantity of matter in a body. Two masses may be compared in a common pair of scales or a balance, because the gravitation force acting on each particle of the two bodies is the same. The ordinary process of weighing is an operation for determining whether two masses are equal, or how many times the one is greater than the other. The accuracy of the method depends on the fact that the masses to be compared are practically at the same distance from the earth's centre.

§ 63. **Gravitation Unit of Force.**—The pound-weight is sometimes taken as a unit of force, and is called the *gravitation* unit, to distinguish it from the absolute unit, which we have already defined (§ 60). If a body whose mass is one pound fall freely for one second it will acquire a velocity of 32·2 feet per second, i.e. the gravitation force between the earth and that mass, which we call its *weight*, will give to the mass of one pound about 32·2 units of velocity in one second. It follows, therefore, that the force of a *pound-weight* can produce in a second about 32·2 units of momentum, and consists of 32·2 units of force. A force equal to $\frac{1}{32 \cdot 2}$ pound-weight is consequently sufficient to produce in one second one unit of momentum, and is equivalent, therefore, to the absolute unit of force or *poundal*. This may be roughly taken as the force of a half-ounce weight (avoirdupois); so that a force equal to a half-ounce weight acting on a mass of a pound for one second will give it a velocity of one foot per second. Similarly, it may be shown that the force equal to the weight of a *gram* is equivalent to about 981 *dynes*, or C.G.S. units of force. These facts, which will later on be experimentally verified, show that the relation between *gravitation* and *absolute* units of force may be expressed by the equation:

$$\text{Gravitation unit} = g \text{ Absolute units.}$$

As the absolute unit of force can be approximately

measured by a certain weight, all forces can be represented by equivalent weights, provided the comparison is made at the same distance from the earth's centre, or with the necessary corrections for the variation in the gravitation force.

§ 64. **Centre of Gravity.**—In connection with the word weight it is well to define a term of frequent occurrence in mechanical problems, and which we shall have afterwards to consider more carefully in its relation to particular bodies. The word 'centre of gravity' is used to express that one point at which the whole weight of a body may be supposed to act. Such a point is found to exist with respect to every body, and, knowing the position of this point, we are able to consider the weight of the body as a force acting, at that point, vertically downwards. Where it is not necessary to consider the pulling force due to the weight of the particles of which the body consists this point is called the *centre of mass*, or *mass-centre.*

§ 65. **Stress and Strain.**—When a solid body is acted upon by *external forces* in such a manner that the body, as a whole, does not move, the shape of the body is usually more or less altered, i.e. the normal position of the particles with respect to one another is changed. We see an example of this when we stretch a piece of india-rubber. In

all bodies this change of the position of the particles, provided it be not so great as to cause rupture or fracture of the body, causes *internal forces* to come into play, tending to restore the body to its original shape. These internal forces are called *stresses*, and the change of shape producing them is called a *strain*. In many mechanical problems the strains with their accompanying stresses are neglected, and bodies subjected to the action of external forces are considered as *rigid*. Thus a perfectly *rigid* body is one the shape of which is not changed, i.e. in which no *strain* is produced by the action of external forces, however great. In nature no such body exists, and although in some cases the strains and stresses may be neglected, and the body may be regarded as rigid, in most practical problems they have to be considered and carefully calculated. These calculations are usually difficult, and belong to a branch of mechanics outside the scope of the present work. These considerations, however, further illustrate the proposition of the introductory chapter of this book, that motion is the universal attribute of matter.

§ 66. **Pressure.**—When a heavy body rests on a hard surface and is prevented from producing motion by the stresses set up among the particles of the substance on which it rests, it is said to exert a *pressure*. Pressure may be produced by

other forces than weight; but in all cases the tendency of the force to produce motion in the mass of a body, as a whole, is counteracted. Since pressure has always reference to forces acting on a surface, its proper measure is the force per *unit of area* of the surface acted upon, and the advantage of this use of the term is clearly seen in treating of the dynamics of fluid bodies.

VIII. *Dynamical Formulæ—Atwood's Machine—Problems*

§ 67. Suppose a force the magnitude of which we will call F is capable of giving to a mass M a velocity of a feet per second during every second of its action, then the force F will generate in one second $M a$ units of momentum, and $M a$ will be the measure of F. Since the unit of force is that force which can produce in one second one unit of momentum, the force F which generates in one second $M a$ units of momentum must contain $M a$ units of force, or

$$F = M a.$$

This is the fundamental proposition of dynamics, and expresses in absolute units the measure of a force.

If we write the equation in the form $a = \frac{F}{M}$, it tells us that, if a force F act on a mass M, the acceleration produced is the ratio of the force causing motion to the mass moved.

§ 68. Suppose W to be the weight of a body having M units of mass at a place where the gravity acceleration is g, then the mass M, if free to fall, will acquire in one second a velocity of g feet per second. Since the weight W is the measure of the force operating between the earth and the body and causing M to fall, W is a force which will generate in one second $M\,g$ units of momentum, and consequently

$$W = M\,g.$$

This equation expresses the relation between the weight of a body and its mass, and shows that the weight of a body equals $M\,g$ absolute units of force. If we substitute the value of M given by the equation $W = M\,g$ in the general equation $F = M\,a$, we obtain

$$F = \frac{W}{g} \cdot a,$$

cr $$F : W :: a : g\,; \text{ or, } a = \frac{F}{W} \cdot g.$$

This last equation is most important in the solution of kinetic problems, and expresses the fact that if a force F act for one second on a certain quantity of matter, the weight of which is W, where the acceleration due to gravity is g, then the velocity generated in every second of F's action is a feet per second, where $a = \frac{F}{W} \cdot g$.

In this equation F stands for the measure of a

definite quantity of any one of the physical forces, and is supposed to be acting constantly for some definite period of time. It may mean a muscular force or the elastic force of steam which produces motion in a variety of ways, or the force of an elastic spring which animates a watch, or a certain amount of heat capable of stretching an iron bar. It may also represent a resistance such as friction, which tends to destroy momentum, and which acts in a direction opposite to that in which motion is about to take place. In all cases, if F be the moving or retarding force, and a the velocity produced or destroyed in one second,

$$F = M a$$

and

$$a = \frac{F}{W} \cdot g,$$

where W is the weight of the mass M.

§ 69. In the foregoing proposition F and W are supposed to be measured in absolute units of force—i.e. in *poundals* or *dynes*—but the proposition is equally true if F and W are measured in gravitation units, provided g is unchanged, since the gravitation unit equals g times the absolute unit of force. It is essential, however, that *both* F and W should be measured *either* in absolute *or* in gravitation units.

Moreover, if P and M be two masses, the weights of which in absolute units are F and W

respectively, then $F = Pg$ and $W = Mg$ (§ 68), and if the action of P's weight on M cause an acceleration a, then $a = \frac{F}{M} = \frac{Pg}{M}$, a form of the equation which is of frequent use, where we are dealing with the acceleration produced by the weight of one mass acting on another mass.

§ 70. The relation that subsists between the quantities of F, W, and a, or between P, M, and a, as given in the above equations, may be verified in many different ways, but most conveniently by a machine called, after its inventor, Atwood's Machine, by means of which we can show that if M remains constant a varies with P, and if P remains constant a varies inversely with M.

§ 71. **Atwood's Machine.**—This consists essentially of a grooved wheel, revolving with as little friction as possible, over which passes a fine thread supporting at one end a mass P, and at the other end a mass Q. The wheel is fixed at a considerable height above the floor, and close to it is fixed a pillar with a scale graduated in centimetres or in feet and inches. To lessen the friction, the axis of the revolving wheel is mounted on the circumference of four smaller wheels. A pendulum beating seconds, or some instrument for accurately marking time, must be used with the machine. If the two masses P and Q are equal they will either remain

at rest or move uniformly with the velocity imparted to them. If a small mass be added to either, the weight of that mass becomes the moving force, and the three masses together represent the whole of the mass moved. The additional mass, the weight of which serves as a moving force, is

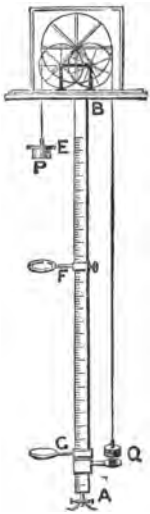

FIG. 19

often in the form of a small bar, so that it may remain on the top of a ring F with which the upright pillar $A\ B$ is furnished, and through which the mass P can pass. If now the ring F be placed at such a distance below the starting-point that the mass P reaches it in one second, the velocity then acquired will equal a, the acceleration ; and if the

bar be left on the top of the ring, the mass will move uniformly with the velocity already acquired. If then a stage G be placed so far below the ring as to stop the mass P in one second after the bar has been moved, the distance FG will measure a, the acceleration. Then it will be found that, if the mass of the bar be increased whilst the whole mass remains the same, the acceleration will vary directly with the force causing motion; and if the mass of the bar remain the same, whilst the equal masses are increased, the acceleration is correspondingly diminished. If, also, the value of a be calculated from the formula $a = \frac{Pg}{M}$, the distance FG will be found to accord with it, and in every other respect, by giving values to the symbols in this expression, the experiment will illustrate the results arrived at in the preceding paragraph.

One advantage of this machine is that the value of a, and consequently the distance FG, can be made as small as we please by taking the mass of the bar sufficiently small compared with the two equal masses. If the two masses P and Q are not equal, then the force causing motion is the difference between the weights of P and $Q = (P - Q)g$. The mass moved is $P + Q$; and

$$\therefore \text{ since } a = \frac{F}{M}, \text{ we have } a = \frac{(P-Q)g}{P+Q}.$$

As soon as the acceleration is determined, all

problems connected with the motion of the masses can be determined by substituting its value for a in the fundamental equations

$$v - u = \pm a t;$$
$$s = \frac{v + u}{2} . t, \text{ or } s = u t \pm \tfrac{1}{2} a t^2;$$
$$v^2 - u^2 = 2 a s.$$

§ 72. **Examples.**—(1) To determine the value of g. Atwood's machine may be employed in determining the value of g.

Thus, suppose the mass of the bar to be 1 oz., and each of the two equal masses to be 15·6 oz., and that in this case the distance $F G$ is found by experiment to be 1 foot.

Then, since $a = \frac{Pg}{M}$, and $P = 1$ oz., and $M = (2 \times 15·6 + 1)$ oz. $=$ 32·2 oz., we have

$$a = 1 = \frac{1}{32·2} . g, \text{ or } g = 32·2 \text{ (feet, secs.)}$$

Again, suppose $P = 50$ grams, and $Q = 48$ grams, and that it is found by experiment that P starting from rest descends through 40 centimetres in 2 seconds. Then from the formula $s = \frac{1}{2} a t^2$ we have $40 = \frac{1}{2} a \times 4$, or $a = 20$;

and since $a = \frac{P - Q}{P + Q} . g$, we have $20 = \frac{2}{98} g$, or

$$g = 980 \text{ (cents., secs.)}$$

(2) To find the distance passed over from the starting-point by either mass in 3 seconds, when $P = 5$ oz. and $Q = 3$ oz.

Here, the force producing motion is $(5 - 3) g =$ 2 oz., weight, and the mass moved is $5 + 3 = 8$ oz.;

$$\therefore a = \frac{2}{8} 32 = 8, \text{ taking } g = 32.$$

$$\text{Also, } s = \frac{a\,t^2}{2} = \frac{3 \times 3^2}{2} = 36 \text{ feet.}$$

(3) A body whose mass is 6 lbs. rests on a perfectly smooth horizontal table and is drawn along by a weight of 2 lbs., attached to it by a string that passes over a smooth pulley at the edge; find the velocity after 4 seconds.

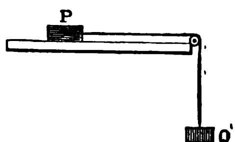

FIG. 20

Since the weight of P is entirely supported by the table, the moving force is the weight of 2 lbs. hanging vertically downwards, i.e. 2 lbs. weight, or $2\,g$, and the mass moved $= 6 + 2 = 8$ lbs.

$\therefore a = \frac{2}{8}\,32 = 8$; and $v = a\,t = 8 \times 4 = 32$ feet per second.

(4) Two masses of 9 oz. and 7 oz. hang over a smooth wheel; motion continues for 5 seconds, when the string breaks; find the height to which the lesser mass will rise after the breakage.

As before, we have

$$a = \frac{\text{moving force}}{\text{mass moved}}$$

$$= \frac{(9-7)\,g}{9+7} = \frac{2}{16} \cdot 32 = 4,$$

$$\text{and } v = a\,t = 4 \times 5 = 20;$$

$\therefore$ each mass has an initial velocity of 20 feet per second when the string breaks;

$$\therefore v^2 = \overline{20}^2 - 2 \times 32 \times s;$$

$\therefore\ s = \frac{400}{64} = 6\frac{1}{4}$ feet, i.e. the lesser mass will rise $6\frac{1}{4}$ feet before it begins to descend.

(5) A steam-engine is moving at the rate of 30 miles an hour when the steam is turned off; supposing the friction to be equivalent to a retarding force of $\frac{1}{400}$ of the weight of the engine, find how long and how far it will move before it stops.

Let W be weight of engine, then if F be the retarding force,

$$F = \frac{W}{400}, \text{ and since } a = \frac{F}{W} g, \text{ we have}$$

$$a = \frac{\frac{W}{400}}{W} g = \frac{g}{400}.$$

The initial velocity is 30 miles an hour, i.e. $\frac{1760 \times 3 \times 30}{60 \times 60}$ $= 44$ feet per second. The question is, In how long a time will this velocity be destroyed, if the velocity be retarded each second $\frac{32}{400}$ feet per second? Since $u = a\,t$, we have

$$44 = \frac{32}{400} t, \text{ or } t = \frac{17600}{32} = 550 \text{ seconds};$$

$$\text{also } u^2 = 2\,a\,s \ \therefore\ 44 \times 44 = \frac{64}{400} s;$$

$$\text{or } s = \frac{44 \times 44 \times 400}{64} = 12{,}100 \text{ feet.}$$

(6) For how long a time must a force of 3 oz. weight act on a mass of 12 oz. to generate a velocity of 40 feet per second?

$$\text{Here } a = \frac{3}{12} 32 = 8, \text{ and } v = a\,t,$$

$$\therefore\ 40 = 8\,t \text{ or } t = 5 \text{ secs.}$$

(7) Equal masses of 500 grams hang from the extremities of a string passing over a smooth pulley. If 20 grams be removed from one of the masses find the distance through which either moves in 3 seconds.

If 20 grams be removed from one of the masses, the moving force is 20 g, and the mass moved is 980 grams;

$$\therefore a = \frac{20}{980} \cdot g,$$

and taking g equal to 980 (centimetres, seconds) we have $a = 20$.

To find the distance from rest in 3 seconds,

$$s = \tfrac{1}{2} a t^2 = 10 \times 9 = 90 \text{ centimetres.}$$

(8) A body weighing 12 oz. is moving along a rough table, on which the friction is equivalent to a force of 3 oz., with a velocity of 20 feet per second. After one second, it reaches the edge of the table and falls to the ground in 2 seconds; find the distance of the point of fall from the edge of the table.

Here the retarding force is equivalent to a weight of 3 oz., and the weight of the mass is 12 oz.,

$$\therefore a = \frac{3}{12} 32 = 8.$$

If v be the velocity after one second, $v = u - at = 20 - 8 = 12$.

The body, therefore, leaves the table with a horizontal velocity of 12 feet per second. It reaches the ground in 2 seconds, in which time it will have fallen through $\frac{32 \times 2^2}{2} = 64$ feet. In 2 seconds it will have moved horizontally $2 \times 12 = 24$ feet.

$\therefore$ distance required $= \sqrt{(\overline{64}|^2 + \overline{24}|^2)} = 68$ feet nearly.

In the preceding examples data have been given respecting the force causing motion and the mass moved, and the first step in our calculations has been to find the acceleration a from the fundamental *dynamical* equation. Any question connected with the motion is then resolved by using the kinematic equations $v - u = a\,t$, &c. In another class of problems the data given are those connected with the motion, and the problem is to find either the force causing motion, or the mass moved, or something on which these depend. In these problems the first step is to find a from the *kinematic* equations, and then by substituting the value of a thus found in the dynamical equation the problem is solved.

(9) A force of 4 lbs. weight causes a certain mass to move from rest through 18 feet in 3 seconds; find the mass.

$$\text{Since } s = \tfrac{1}{2}\,a\,t^2, \text{ we have } 18 = \frac{9\,a}{2} \therefore a = 4.$$

$$\text{Also } a = \frac{F}{M}, \text{ or } 4 = \frac{4 \times 32}{M} \therefore M = 32 \text{ lbs.}$$

(10) A mass of 50 lbs. is acted on by a constant force which acts for 5 seconds and then ceases to act; the body moves through 60 feet in the next 2 seconds. Express the force in absolute units.

In this question, when the force ceases to act the body moves uniformly through 60 feet in 2 seconds, and $\therefore$ the velocity produced by the action of the force is 30 feet per second; and since this velocity is gained in 5 seconds, the acceleration, or increase of velocity per second is 6. Taking $M = 50$, and $F = M\,a$, we have $F = 6 \times 50 = 300$ *poundals.*

(11) What is the measure of the force equivalent to

the weight of 3 kilograms at a place where a body falls freely through 44·1 metres in 3 seconds?

$$\text{Here } s = \tfrac{1}{2} g t^2;$$

$$\therefore 44{\cdot}1 = \frac{g}{2} \times 9 \therefore g = \frac{2 \times 4410}{9} = 980 \text{ (cents., secs.)},$$

and since $W = Mg$, we have weight of 3 kilograms = $3 \times 1000 \times 980$ (grams, cents., secs.) = 2,940,000 dynes.

(12) A plane supporting a weight of 12 lbs. is descending with a uniform acceleration of 10 (feet, secs.); find the force that the weight exerts on the plane.

When a mass rests on a horizontal surface it exerts a downward force equal to its own weight. If the surface move downward with an acceleration of 32·2, the mass exerts no force whatever. This would be the case if we were to place one book on another, and then let the lower book fall. During the motion, the upper book would exert no force on the lower book, since they would both move together.

Let F be the force with which the body in the question presses on the plane when moving vertically downwards with an acceleration of 10. Then the force causing motion is the difference between the weight and the force with which the mass presses on the plane $= 12g - F$, and the mass moved is 12, and $a = 10$; $\therefore 10 = \dfrac{12g - F}{12}$, and taking $g = 32$ we have $F =$ 264 *poundals*.

The force with which the mass presses on the plane is, therefore, 264 *poundals*, and is equal to the weight of $\dfrac{264}{32} = 8\tfrac{1}{4}$ lbs.

The next two examples belong to the first class of problems, but refer to motion on an inclined plane.

(13) A mass Q supported on an inclined plane, which rises h in l, is pulled up the plane by the weight of a mass P connected with Q by a thread, which passes over a wheel at the summit of the plane, and hangs vertically downwards. Find the acceleration on the plane.

Let F equal the force with which the body, whose

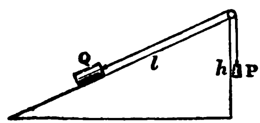

FIG. 21

mass is Q, is urged down the plane, and a its acceleration in that direction, then

$$a = \frac{F}{Q}.$$

$$\text{But } \frac{a}{g} = \frac{h}{l} \text{ (§ 50) } \therefore F = \frac{h}{l} Q g.$$

If, therefore, the weight of P cause Q to *ascend* the plane, the force causing motion must be $P g - F = (P - \frac{h}{l} Q) g$, and the acceleration $a = \dfrac{P - \frac{h}{l} Q}{P + Q} \cdot g.$

(14) Two masses P and Q are supported on two inclined planes, the lengths of which are l and l' and the common height h. They are connected by a fine thread as before; find the acceleration.

If F equal the force with which P is urged down the plane, then $F = \frac{h}{l} P g$, as above, and if F' equal the force

with which Q is urged down the plane, then $F' = \frac{h}{l} Q\ g$, and if P is able to draw up Q, the force causing motion is $F - F' = \frac{h}{l} P g - \frac{h}{l'} Q g$, and the acceleration

$$a = \frac{h}{l\,l'} \cdot \frac{P l' - Q l}{P + Q} \cdot g.$$

(15) A body is projected up an inclined plane which rises 1 in 10 with a velocity of 40 feet per second. Supposing the effect of friction to be equivalent to a uniformly retarding force equal to $\frac{1}{100}$ the weight of the body, find how far the body will move up the plane.

Here there are two causes tending to lessen the velocity of projection: first, gravity; secondly, friction.

Retardation due to gravity $= \frac{h}{l} g = \frac{1}{10} g$;

" " " friction $= \dfrac{\frac{W}{100}}{W} \cdot g = \frac{g}{100}$, where W is the *weight* of the body;

$\therefore$ resultant retardation equals $\left(\frac{1}{10} + \frac{1}{100}\right) g$.

Since $v^2 = 2 a s$ we have

$$(40)^2 = 2 \left(\frac{1}{10} + \frac{1}{100}\right) g . s$$

$$= \frac{22}{100} . g . s,$$

$$\therefore s = 227 \frac{3}{11} \text{ feet.}$$

If we had taken M to be the mass of the body $M g$ would have been the weight of the body, and the retardation due to friction would have been

$$\frac{\frac{M g}{100}}{M} = \frac{g}{100}, \text{ as before.}$$

EXERCISES VII

1. Find the measure of the force that must act on a mass of 10 kilograms, moving with a velocity of 15 metres per second, to bring it to rest in 5 minutes.
2. In Atwood's machine, if the two masses are 119 grams and 121 grams, and the distance passed over in the third second is 8 inches: find the value of g.
3. Find the force which, acting on a mass of 3 kilograms for $4\frac{1}{2}$ seconds, gives it a velocity of 23 metres per second. $(g = 980.)$
4. A force F acting on a mass of 5 lbs. increases its velocity every second by 12 feet per second; another force F' acting on a mass of 28 lbs. increases its velocity in every second by $7\frac{1}{2}$ feet per second; find the ratio of F to F'.
5. A mass of 200 grams is acted on by a force equal to the weight of 10 grams for 20 seconds. What distance will the mass have passed through, and what velocity will it have acquired? $(g = 980.)$
6. Two bodies the masses of which are as 3 : 2 are found by a spring balance to weigh in two different places 963 grams and 628 grams respectively. Compare the velocities acquired by a body falling for 1 second in each of the two places.
7. If the unit of length were a yard and the unit of time were a minute, how would the number expressing the value of g be correspondingly changed?
8. Two masses 15·25 and 16·75 hang over a pulley; find their velocity after 4 seconds.
9. Find in what time a force[1] of 5 lbs. will move a mass of 16 lbs. through 45 feet along a smooth horizontal plane.

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[1] A *force* of 5 lbs. means a force of 5 lbs. weight, and produces the same effect as a weight of 5 lbs. acting vertically downwards; its equivalent is 5 g *poundals*.

10. Find in what time a mass weighing 5 lbs. hanging vertically downwards will move a mass of 11 lbs., with which it is connected by a string passing over a pulley through 45 feet along a smooth horizontal plane.

11. Two bodies whose masses are m and $3m$ are moved by forces $3p$ and p respectively; compare the distances they pass over in t seconds.

12. What force must act on a mass of 48 lbs. to increase its velocity from 30 to 40 feet per second, whilst it passes over 80 feet?

13. A steam engine is moving on a level road at the rate of 30 miles an hour; find the ratio of the brake power to the weight of the engine, if the engine is brought to rest in 200 yards.

14. If the weight of a certain mass is 10 at a place where a body falls through 64 feet in 2 seconds, what will be the weight of the same mass at a place where a body falls through 101·25 feet in 2·5 seconds?

15. What mass hanging vertically downwards will draw a mass of $3\frac{1}{2}$ lbs. across a perfectly smooth table 8 feet wide in 2 seconds?

16. Through what distance must a force of 3 lbs. act on a mass of 16 lbs. to give it a velocity of 6 feet per second?

17. Two masses hang over a pulley; the greater weighs 12 oz., and it moves the smaller through 36 feet in 3 seconds; find the smaller.

18. What is the force of friction if a body whose mass is 20 lbs. projected along a rough horizontal plane with a velocity of 48 feet per second, come to rest after 5 seconds?

19. What mass must be added to one of two equal masses of 6 oz. which hang over a smooth wheel so that they may move through 16 feet in 5 seconds?

20. Required the force that will cause a mass of 25 lbs. to move through 320 feet in 10 seconds.

21. The velocity of a body of mass 12 lbs. increases from 10

feet per second to 20 feet per second whilst the body passes over 15 feet; what is the moving force?

22. Two masses, one of which is 4 oz., hang over a pulley; the mass of 4 oz. ascends with a uniform acceleration of 5 (feet, seconds). Find the other mass.

23. Two masses of $7\frac{3}{4}$ oz. and $8\frac{1}{4}$ oz. connected by a thread hang over a pulley; motion continues for 3 seconds, when the string breaks. To what height will the smaller mass ascend, and how far will the larger mass fall in 4 seconds after the breaking of the string?

24. If we take a pound as unit of mass, and a foot-second as unit of velocity, how many units of force acting for 2 seconds will be required to give a mass of 10 lbs. a velocity of 50 feet per second?

25. Two masses, of $7\frac{1}{4}$ and $8\frac{3}{4}$, hang over the wheel of Atwood's machine, and are set in motion from the same horizontal level; how far will they be apart in $2\frac{1}{4}$ seconds?

26. For how long a time must a force of 1 lb. weight act on a mass of 40 lbs. to give to it a velocity of 500 feet per second?

27. A mass of 100 lbs. is moving horizontally with a velocity of 20 feet per second, and is retarded by friction which is equivalent to a force of 5 lbs. How far will it move?

28. To one end of a string hanging over a pulley is attached a mass of 5, and to the other end two masses of 3 and 4. Motion takes place for 3 seconds, when the mass of 4 is removed; for how long will the mass of 5 continue to ascend?

29. Two masses P and Q hang over a smooth wheel, connected by a string, and P descends 18 feet in 3 seconds. If, however, 5 oz. had been added to Q, Q would have descended through the same space in $4\frac{1}{2}$ seconds; find P and Q.

30. A steam engine is moving at the rate 20 miles an hour when the steam is shut off; if the force of friction be equivalent to $\frac{1}{320}$ of the weight of the engine, after what time will it stop?
31. A plane sustaining a weight of 20 lbs. is descending with a uniform acceleration of 10 (feet, secs.); find the force pressing on the plane.
32. A body weighing 1 cwt. goes up and down on a lift with a uniform acceleration of 4 (feet, seconds); find the force pressing on the lift in each case.
33. A mass of 20 oz. is moved on a smooth horizontal table by a weight of 4 oz. connected with it by a string which passes over a pulley at the edge of the table and hangs vertically downwards. After 3 seconds the mass of 20 oz. reaches the edge of the table, and the string breaks. At what distance from the top of the table will it strike the ground, supposing it to reach the ground in 1 second?
34. If a force of 6 lbs. gives to a certain mass a velocity of 20 feet per second in 4 seconds, what velocity will a force of 4 lbs. give to twice the mass in 6 seconds?
35. For how long a time must a force of 3 lbs. act on a mass of 120 lbs. to give to it a velocity of 960 yards per minute?
36. The last carriage of a railway train gets loose whilst the train is running at the rate of 30 miles an hour up an incline of 1 in 150. Supposing the effect of friction upon the motion of the carriage to be equivalent to a uniformly retarding force equal to $\frac{1}{300}$ the weight of the carriage, find, first, the length of time during which the carriage will continue running up the incline, and, secondly, the velocity with which it will be running down after the lapse of twice this interval from the instant of its getting loose.
37. A body weighing 6 lbs. is drawn up along the lid, 4 feet long, and rising 2 in 9, of a smooth desk by a

weight of 5 lbs. which, attached to the mass of 6 lbs. by a string, hangs over the top of the desk and descends vertically. Find the velocity acquired when the 6-lb. mass reaches the top of the desk.

38. If the mass 10 lbs. be the unit of mass, and 1 yard and 1 minute be the units of length and time, how would you define the unit of force? and how many such units of force are there in the weight of 3 tons?

39. Two planes, inclined at one-third and two-thirds of a right angle to the horizon respectively, meet at the top and slope opposite ways. If bodies of equal mass fall down these planes, starting at the same instant, find their accelerations.

40. The weights at the extremities of a string which passes over the pulley of an Atwood's machine are 500 and 502 grams. The larger weight is allowed to descend, and 3 seconds after the motion has begun 3 grams are removed from the descending weight. What time will elapse before the weights are again at rest?

IX. *Impulsive Forces—Impulse*

§ 73. **Impulse.**—The forces we have hitherto considered have been such as, acting for a second of time at least, produce a certain momentum or change of momentum, by which they have been numerically expressed. These forces are sometimes called *moving forces*—a term which expresses no more than force itself, since all forces necessarily tend to produce motion—they are *constant* or *uniform* forces. We have now to consider a class of forces which continue to act for an almost

infinitely short space of time, and cease to act as soon as they have imparted a certain velocity to the mass which they set in motion. We have instances of the effect of such forces when a ball is thrown from the hand, or struck by a bat, when an arrow is shot from a bow or a bullet from a gun, when a nail is hammered into the wall, or when piles are driven into the ground. These forces are called *impulsive forces*, and it is clear we cannot measure their magnitude by the momentum produced in a second, since their action is limited to a much shorter period of time. Their effect can, however, be measured by the total change of momentum produced, and this new momentum will remain constant unless changed by some other force. Thus if a bat strike a moving ball, the momentum of the ball is changed by the blow in the immeasurably short space of time that the bat is in contact with the ball. As a fact the particles of both bodies undergo considerable displacement in this short interval of time, but the resultant or aggregate effect of the internal stresses called into action is the change of momentum produced in the two bodies brought into momentary contact. This effect is called an *impulse*.

§ 74. **Measure of Impulse.**—We have seen that any constant force is measured by the change of momentum, in one second, of the mass it acts

upon, or, more briefly, by the *rate* of change of momentum. But the effect of a constant force acting for t seconds on a mass M is to produce a total change of momentum which is measured by $F t = M a t = M v$, where $F = M\ a$ is the change of momentum in one second.

Now $F t = M v$ is equally the measure of the effect of an impulsive force, or more correctly of the impulsive forces, which produce a *blow* or *impulse*; but in this case t, instead of being any definite period of time, is a period of time too small to be measured. As a fact F, which represents the molecular forces called into action by the impact, is continuously changing even throughout this small interval of time, rising and falling in value. The aggregate effect, however, of the action of F throughout the small period of time is to produce a momentum $M v$ on the mass acted upon, that is, to cause a body whose mass is M to move with a velocity v, which velocity remains constant with the cessation of the action of the force. If we call P the measure of the effect of these impulsive forces acting for an indefinitely short period of time $P = M v$, and this equation is analogous to the fundamental equation $F = M a$, the difference being that in the case of the impulsive forces the velocity v is generated in a time t too small to be measured, and then remains constant; and in the

case of the constant force the velocity a is produced in one second, and the same amount of velocity is given to the mass M during every second of the action of the force. An impulse is always measured in units of momentum.[1]

With this explanation we may use the equation $P = M v$ to measure the effect of an impulsive force and to represent the relation between the forces and the change of momentum they produce.

§ 75. **Examples.**—(1) If a hundred-pound shot projected from a gun ascend vertically for 3 seconds, through what height will a fifty-pound shot ascend if projected with three times the charge?

Let P be the impulse given by the powder in the first case, and let P' be the impulse in the second case.

Then if u be the velocity with which the shot leaves the gun in the first case, and u' the corresponding velocity in the second case,

$$u = \frac{P}{100}, \text{ and } u' = \frac{P'}{50}$$

But $P' = 3 P$ supposing the impulse proportional to the charge. $\therefore u' = \frac{3 P}{50}$

and $\frac{u}{u'} = \frac{1}{6}$.

[1] It has been proposed to call the unit of momentum a *poundem* or a *grammem*, according to the unit of mass adopted.

Now, since the shot ascends in the first case for 3 seconds, $u = 3\,g$;

$\therefore u' = 6\,u = 18\,g = \sqrt{2\,g\,h}$, where h is the height required;

$\therefore h = 5{,}184$ feet nearly.

(2) A shot of 2 lbs. mass projected from a gun rises to a height of 100 feet; to what height will a shot rise of 5 lbs. mass, if projected with three times the charge?

Using the same symbols as in the last example, we have

$$u = \frac{P}{2},\ u' = \frac{P'}{5},\ \text{and}\ \frac{P}{P'} = \frac{1}{3}.$$

$$\therefore \frac{u}{u'} = \frac{5}{6},$$

and
$$\frac{u^2}{u'^2} = \frac{25}{36} = \frac{2\,g\,h}{2\,g\,h'}.$$

$$\therefore \text{since } h = 100,\ h' = 144 \text{ feet.}$$

§ 76. **Equivalent Constant Force.**—If the distance through which the body moves whilst under the influence of the impulsive forces is given, the average constant force which under the same circumstances would have produced the same effect can be calculated. For instance, if we know the length of the bore of a gun and can find the velocity with which the shot leaves the muzzle, we can find the constant force which, acting through the whole length of the bore, would have produced the same effect. Or, again, if we know the impulse with which a shot strikes a wall and the

distance it penetrates, we can find the constant force which would have produced the same effect, or the uniform resistance which the wall offers to the passage of the ball.

§ 77. **Example.**—A shot whose mass is 112 lbs. is fired from a fixed gun with a muzzle velocity of 1,500 feet per second, and travels over a distance of 16 feet whilst in the gun; find the measure of the impulse on the shot, and the uniform force which, acting inside the gun, would have given it the same velocity.

The impulse $= Mv = 112 \times 1500 = 168000$.

To find the equivalent force we note that a velocity of 1,500 is acquired in passing over 16 feet, and since

$v^2 = 2as$, we have

$$(1500)^2 = 2a \times 16, \text{ or } a = 70312{\cdot}5.$$

If F be the measure of the uniform force,

$$F = Ma = 112 \times 70312{\cdot}5 = 7{,}875{,}000 \text{ poundals.}$$

Or, taking $g = 32$, the force equals $7{,}875{,}000 \div 32$ $= 246{,}094$ lbs. weight nearly.

Problems of this kind involve the consideration of what is meant by *work* and *energy*, and will be better understood after an explanation of Newton's third law of motion.

EXERCISES VIII

1. An arrow shot from a bow starts off with a velocity of 120 feet per second; with what velocity will an arrow of twice the mass leave the bow, if sent off with an impulse 3 times as great?
2. If a ball of mass $2\,m$ fired from a gun rise to a height of 150 feet, to what height will a ball of mass $3\,m$ rise, if fired with twice the charge of powder?

3. Two balls whose masses are 8 oz. and 6 oz. respectively are simultaneously projected upwards, and the former rises to a height of 324 feet and the latter to 256 feet; compare the impulses with which they are projected.
4. What is the measure of the blow given to a ball weighing 1 lb. if it start off with a velocity of 224 feet per second?
5. Two balls whose masses are as 2 : 3 are projected vertically upwards with the same force; compare the heights to which they rise.
6. A cannon-shot of 1,000 lbs. strikes directly a target with a velocity of 1,500 feet per second, and comes to rest; what is the measure of the impulse?
7. A hammer whose mass is 3 lbs. strikes a nail with a velocity of 20 feet per second, and drives it 1 inch into a board; find the average resistance of the board.
8. A mass of 10 lbs. falls 100 feet, and is then brought to rest by penetrating 1 foot into the sand; find the average resistance on the sand.

EXAMINATION QUESTIONS III

1. A mass originally at rest is acted on by a force which in $\frac{1}{308}$th of a second gives to it a velocity of $5\frac{1}{4}$ inches per second; show what proportion the force bears to the weight of the mass. ($g = 32{\cdot}2$.)—*Univ. of Lond. Matric.* Jan. 1871.
2 If a particle moves in consequence of the continued action upon it of a constant force, show what is the character of the resulting motion, and in what manner it depends on the magnitude of the force and the mass of the particle.—*Ib.* Jan. 1872.
3. As a special case show how the resulting motion would be changed if the mass of the particle were

trebled, and the intensity of the force acting upon it were doubled.—*Ib.* Jan. 1872.

4. The speed of a railway train increases uniformly for the first 3 minutes after starting, and during this time it travels 1 mile. What speed (in miles per hour) has it now gained, and what distance did it traverse in the first 2 mins.?—*Ib.* June 1872.

5. In the last question, supposing the line level and disregarding friction and the resistance of the air, compare the force exerted by the engine with the weight of the train.—*Ib.* June 1872.

6. Which could you throw further, a small ball of lead or a ball of cork of the same size? and why?—*Ib.* Jan. 1873.

7. Find the 'tension' on a rope which draws a carriage of 8 tons weight up a smooth incline of 1 in 5, and causes an increase of velocity of 3 feet per second. —*Ib.* June 1873.

8. If, on the same incline, the rope breaks when the carriage has a velocity of 48·3 feet per second, how far will the carriage continue to move up the incline?—*Ib.* June 1873.

9. When a body changes its rate of motion under the action of a constant force, show that the space described in any time is the same as the space described by a body moving uniformly with the mean velocity for the same time.—*Ib.* Jan. 1874.

10. I suddenly jump off a platform with a 20-lb. weight in my hand. What will be the pressure of the weight upon my arm while I am in the air? Give a reason for your reply.—*Ib.* Jan. 1874.

11. In Atwood's machine one of the boxes is heavier than the other by half an ounce. What must be the load of each in order that the overweighted box may fall through 1 foot during the first second?—*Ib.* Jan. 1874.

12. Two masses of 48 and 50 grams respectively are attached to the string of an Atwood's machine, and,

starting from rest, the larger mass passes through 10 centimetres in 1 second. Determine from these data the value of the acceleration due to gravity, your units being centimetres and seconds.—*Ib.* Jan. 1876.

13. There are two chambers, one of which is in the act of dropping freely under gravity down a pit, while the other is made to descend with uniform velocity. A man in each chamber during the descent lets go a stone which he has been holding in his hand. What will be the motion of the stone in each case? —*Ib.* Jan. 1877.

14. A mass of 488 grams is fastened to one end of a cord which passes over a smooth pulley. What mass must be attached to the other end in order that the 488 grams may rise through a height of 200 centimetres in 10 seconds? ($g = 980$.)—*Ib.* June 1878.

15. A mass of 6 oz. slides down a smooth inclined plane whose height is half its length, and draws another mass from rest, over a distance of 3 feet in 5 seconds, along a horizontal table which is level with the top of the plane, the string passing over the top of the plane. Find the mass on the table. —*Ib.* Jan. 1879.

16. A smooth inclined plane, whose height is one half of its length, has a small pulley at the top, over which a string passes. To one end of the string is attached a mass of 12 lbs., which rests on the plane; while from the other end, which hangs vertically, is suspended a mass of 8 lbs.; and the masses are left free to move. Find the acceleration and the distance traversed from rest by either mass in 5 seconds.—*Ib.* Jan. 1881.

17. Two bodies whose masses are 31 oz. and 33 oz. respectively, suspended at the top ends of a thin string passing over a smooth pulley, are allowed to move freely for three seconds. What will be

the velocity acquired, and what will be the distance traversed by each body?—*Ib.* Jan. 1882.

18. Two scale-pans, each weighing 2 oz., are suspended by a weightless string over a smooth pulley. A mass of 10 oz. is placed in one and 4 oz. in the other. Find the 'tension' of the string and the pressure on each scale-pan.—*Ib.* June 1883.

19. A certain force acting on a mass of 10 lbs. for 5 seconds produces in it a velocity of 100 feet per second. Compare the force with the weight of 1 lb., and find the acceleration it would produce if it acted on a ton.—*Ib.* June 1883.

20. The horizontal and vertical components of a certain force are equal to the weights of 5 lbs. and 12 lbs. respectively. What is the magnitude of the force? Supposing this force to act for 10 seconds on a mass of 8 lbs., which is also exposed to the action of gravity and is initially at rest, what velocity will be communicated to the mass, the vertical component of the force acting upwards?—*Ib.* June 1883.

21. A mass of 19 lbs. and a mass of 5 lbs. are connected by a string which passes over a pulley at the edge of a horizontal table, so that the smaller mass hangs vertically, and, by its weight, pulls the larger mass along the table. Determine the acceleration, friction being neglected.—*Ib.* Jan. 1885.

22. While a railway train travels half a mile on a level line, its speed increases uniformly from 15 miles an hour to 30 miles an hour; show what proportion the pull of the engine bears to the weight of the train [neglect friction].—*Ib.* Jan. 1885.

23. A ball whose mass is 3 lbs. is falling at the rate of 100 feet per second. What force, expressed in pounds weight, will stop it (1) in 2 seconds, (2) in 2 feet?—*Ib.* June 1886.

24. If a force equal to the weight of 10 lbs. act upon a mass of 10 lbs. for 10 seconds, what will be the momentum acquired?—*Ib.* Jan. 1887.

25. A railway train, whose mass is 100 tons, moving at the rate of a mile a minute, is brought to rest in 10 seconds by the action of a uniform force. Find how far the train runs during the time for which the force is applied.—*Ib.* Jan. 1888.

26. A bullet of mass 1 oz. leaves the muzzle of a gun 3 feet in length with a velocity of 1,000 feet per second. Find the average pressure of the powder on the bullet.—*Ib.* June 1889.

27. A body falling freely acquires in 1 second a velocity of 981 c.m. per second. If a force equal to the weight of 1 gram pull a mass of 1 kilogram along a smooth level surface, find the velocity when the mass has moved 1 metre.—*Ib.* June 1890.

28. Does the rope of a colliery hoist have to bear more strain when the cage is at the top or at the bottom of the shaft? To eliminate the weight of the rope itself, consider only the portion immediately above the cage. Explain under what circumstances the stress may be greater than the weight of the cage attached to it.—*Ib.* Jan. 1891.

29. There are two bodies whose masses are in the ratio of 2 to 3, and their velocities in the ratio of 21 to 16; what is the ratio of their momenta? If their momenta are due to forces P and Q acting on the bodies for equal times, what is the ratio of P to Q? —*S. & A. Dep.*, 1890.

30. Equal forces act on two bodies whose masses are M and m; at the end of a second, the former is moving at the rate of 10 miles an hour, and the latter at the rate of 110 feet per second. Find the ratio of M to m.—*Ib.* 1891.

CHAPTER IV

NEWTON'S LAWS OF MOTION

X. *The First and Second Laws*

§ 78. Three laws have been enunciated by Newton as the principles in accordance with which motion takes place, and have generally been made the basis of Dynamical Science. In the preceding pages we have assumed some of the principles involved in these laws, but there are others of great importance which we shall now be able to consider. These laws are the following :—

§ 79. **Law I.**—*Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it is compelled by impressed forces to change that state.*

This law, as founded on experience, is supported by negative evidence only. It tells us that matter is that which possesses inertia, and has been called the Law of Inertia. It consists of two distinct parts. The first part asserts that a body in a state of rest will continue in that state unless

made to move by external force. Now, we have already seen that absolute rest nowhere exists, and that what we call rest is a complex state resulting from the combined action of several forces. Since there is no matter without force and no force without the tendency to motion, the state of rest cannot be regarded as the normal condition of bodies. What this law, therefore, asserts is, that when a body is maintained in a state of rest by the combined action of two or more forces, that state of rest will be continuously preserved unless some other force change the conditions.

The second part of the law asserts what we have already been compelled to assume, that all bodies tend [1] to move uniformly, and in a straight line; that it is not *motion* but *change* of motion which is produced by force, and that the forces which change a body's state of rest or motion are 'impressed,' i.e. *external* forces, and not *stresses* or forces between the parts of a body. In other words, a body once in motion tends to preserve that condition, and to maintain its original

[1] By introducing some word signifying 'tendency' in the propositions of dynamics, we save the necessity of providing for cases in which other forces change the motion of the body. Thus it is incorrect to say that all bodies fall to the earth, because a balloon contradicts this law; but it is perfectly correct to say that all bodies *tend* to fall to the earth, and to this law the motion of the balloon forms no exception.

momentum. In support of this law, which observation abundantly verifies, an appeal is made to the experience acquired by gradually removing the several external causes, which practically, to a greater or less extent, always impede motion. Thus it is found that the smoother the road along which a stone is rolled the further the stone will roll; that the more the air is removed from a chamber in which a pendulum is suspended, the longer it will continue to oscillate; that in all cases where a body once in motion is gradually brought to rest, some external force, such as friction, is known to exist, and that as this force is diminished the duration of the motion in increased. This law asserts not only that the speed or rate of motion will be preserved, but likewise the direction of the motion; and that a body cannot move otherwise than in a straight line except by the continuous action of some external cause.

§ 80. **Law II.**—*Change of motion is proportional to the impressed force, and takes place along the straight line in which that force is impressed.*

This law asserts that whatever motion (and by motion is here understood quantity of motion or momentum) a certain force produces, double as much motion will be produced by twice the force, three times as much by a force three times as great, and so on. It also implies that if a certain

force acting for one second can produce a certain amount of motion, the same force acting for two seconds will produce twice that quantity. This explains how it is that a constant force such as gravity produces an accelerative effect, adding on a fixed increment of momentum every second to the falling body. It also shows us how forces may be measured, for it tells us that equal changes of momentum are due to equal forces acting for equal times, and hence equal forces are such as, applied to the same mass, generate equal increments of speed in the same time. This law is really the fundamental principle of mechanical science; for, since change of motion is proportional to the moving force, when several forces act together the change of motion due to each is proportional to each, and this is the principle which underlies all rules for the composition of mechanical forces. This principle may be enunciated thus :—

When several forces act simultaneously on a body, each produces the same effect as if it had acted separately.

The operations of this law have been already considered in Lessons I. to VI. on Kinematics ; but quantity of motion was there understood to mean velocity only, since the mass of the body was not taken into account. This law includes, therefore, as a special case, the law of the composition of

velocities, already referred to. One consequence of this law is that a given force produces the same effect upon a body, whether the body be at rest or in motion. It also follows that the resultant of any number of forces may be found by the same process as the resultant of any number of velocities, and thus the parallelogram and polygon laws are found to apply to forces as well as to velocities.

The law is illustrated by a variety of phenomena. A ball thrown vertically upwards from the hand of a person in motion will return to him just the same as if he had been stationary: a stone dropped from the top of the mast of a ship falls at the foot of the mast, whether the vessel be sailing or at rest: a body projected horizontally from the top of a cliff reaches the ground at the same point as if it had first moved horizontally with the velocity of projection, and then vertically downwards under the action of gravity for the whole time of flight. Other consequences of this law have been already considered in Chapter I., and further illustrations will be found in the section on the Composition of Forces, Chapter VII.

§ 81. **Motion in a Circle.**—An illustration of this law not yet considered is the case of a body moving with a uniform acceleration towards a

fixed point and with a uniform velocity in a direction at right angles to the direction of the acceleration.

Let P and Q be two positions of the body. Then at P and Q the body is tending to move with a uniform acceleration towards O, and a

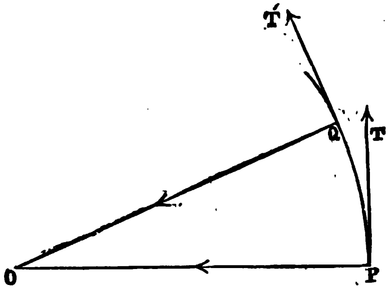

FIG. 22

uniform velocity at P along PT and at Q along QT'.

Now it is evident that the real path of the body will be a curve of which PT and QT' are tangents, and, since the tangential velocity is uniform at all points in the path, the body will move along a circle of which O is the centre with a uniform velocity.

Suppose v to be the constant tangential velocity and a the acceleration towards the centre, and PQ to be the path described in a *very* small time

t. Join PQ and draw QM perpendicular to OP, and join QA.

Then the arc $PQ = vt$, and, since PM is the distance traversed in the direction of the acceleration a,

$$PM = \tfrac{1}{2} a t^2.$$

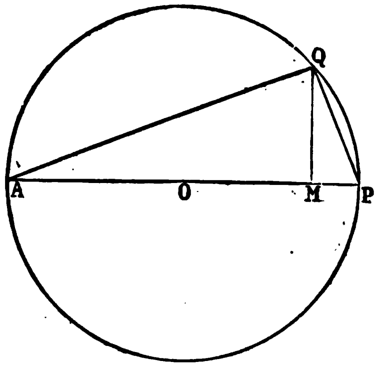

FIG. 23

And by taking PQ small enough, the arc PQ may be supposed equal to the chord PQ,

$$\text{and } \frac{PM}{PQ} = \frac{PQ}{PA} \text{ by similarity of triangles;}$$

$$\therefore\ PQ^2 = PM \times PA, \text{ or } v^2 t^2 = \tfrac{1}{2} a t^2 \times 2r,$$

where r is the radius of the circle.

Hence $a = \dfrac{v^2}{r}$, which is the value of the acceleration towards the centre.

§ 82. **Centripetal and Centrifugal Forces.**—When a body of mass m moves in a circle of radius r with a uniform velocity v, owing to the action of a force F directed towards the centre, the value of F is $m\,a = \frac{m\,v^2}{r}$, and the force is called the *centripetal force.*

The equal force which a revolving body exerts in the opposite direction, and by virtue of which it tends to fly *away* from the centre, is called the *centrifugal force.*

Now since $F = \frac{m\,v^2}{r}$ it is clear that F increases more rapidly with any increase of v than it diminishes with any corresponding increase of r. Where the *angular* velocity of several particles is constant whilst the *linear* velocity varies, the centrifugal force may be shown to vary directly with the radius.

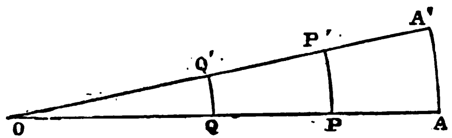

FIG. 24

Thus suppose the line $O\,A$ (fig. 24) revolves about O; the linear velocity of any particle at P is greater than at Q, and is greatest at A, but the angular velocity of all points along $O\,A$ is the same.

Let ω = this angular velocity. Then, since $v = r\,\omega$ (§ 9), we have $F = \frac{m\,v^2}{r} = \frac{m\,r^2\,\omega^2}{r} = m r\,\omega^2$; which shows that for the same angular velocity F varies directly with r. If we apply this to the motion of particles on the surface of the earth we shall see that F is greatest at the equator, and diminishes as we pass from the equator to the poles.

Here we have a further reason for the fact that the value of force of gravitation, and consequently of g, is least at the equator and greatest at the poles. For gravity is diminished by the whole of the centrifugal force at the equator, and at the poles this force is zero. Moreover, whilst at the equator the centrifugal force acts in a direction exactly opposite to that of gravity, the line of action of this force as we approach the poles is inclined at an increasing angle to that of gravitation; and consequently not only is the force itself *less*, but gravity is diminished by a part only of its value.

XI. *The Third Law of Motion*

§ 83. **Law III.**—*To every action there is an equal and contrary reaction; or the mutual actions of two bodies on one another are always equal and in opposite directions.*

This law asserts that if A exerts a force on B, B exerts an equal force in the opposite direction upon A; that every force is *one* of a pair of forces. Such a pair of forces is termed a *stress.* The *stresses* between the different parts of a body or system are called *internal* forces. A force which acts on a body, and which is one of the pair of forces forming a stress between it and another body, is an 'impressed' or external force. Such a pair of forces are the centrifugal and centripetal forces considered in the last lesson. This law is illustrated by a variety of phenomena.

§ 84. **Statical Reaction.**—If one body presses another it is at the same time equally pressed by the other body, but in an opposite direction. If a heavy body rest on a hard surface it presses against that surface, and is, at the same time, equally pressed by the surface in the opposite direction. This opposing pressure of the hard surface is called *Statical Reaction* and is always perpendicular to the surface, and exactly equal to the pressing force which the body causes in that direction. This is true whether a body rests on a surface that is horizontal or inclined at an angle to the horizon. In the former case the reaction is equal to the whole weight of the body; but in all other cases it is equal to the efficient pressing force exerted, i.e. to the resolved part, according to the

parallelogram law, of the weight which acts in a direction perpendicular to the surface. The weight, or the resolved part of it, and the reaction form the stress or pair of forces acting between the heavy mass and the surface of the body supporting it.

§ 85. '**Tension,' Pull.**—In the case of a stretched cord we have another instance of the fact that every force is one of a pair of equal forces acting in opposite directions. If a horse draw a tram car by means of a rope, the horse is drawn in the opposite direction by a force equal to that exerted by the horse through the rope. To the force acting through the stretched cord the name 'tension' has usually been given; but the word 'tension' has acquired in Hydrostatics a distinct and definite meaning, and we shall use the word *pull* to designate a force acting through a stretched string. That a *pull* is a force which acts equally in both directions, and at any point of a stretched string, may be seen if a string be attached to a fixed wall and stretched, and afterwards removed from the wall and stretched in the opposite direction by a force sufficient to keep the original force in equilibrium. If the string be cut at any point, a force equal and opposite to the stretching force would similarly be required to preserve equilibrium, showing that the stretching

force, or *pull*, is the same at all points of the string. The string along which the pull acts is supposed to be perfectly flexible, inextensible, and weightless.

§ 86. **Example.**—Two masses m and m' hang by a flexible thread over a smooth wheel, as in Atwood's machine, and m is greater than m'; what is the force by which the string is stretched?

Let T equal the stretching force. We must consider the motion of each mass separately. Since m is greater than m', the mass m will move downwards and m' upwards.

The mass m is pulled upwards by the stretching force T, and downwards by the weight mg. The force causing motion in the mass m is $mg - T$, and

$$\therefore a = \frac{mg - T}{m};$$

and since the pull is the same throughout the whole length of the string, and the mass m' ascends with the same acceleration as the mass m descends, both being connected by a string, $T - m'g$ is the force causing the mass m' to ascend. Hence

$$a = \frac{T - m'g}{m'}$$

and

$$\frac{mg - T}{m} = \frac{T - m'g}{m'},$$

or

$$T = \frac{2mm'g}{m + m'} \text{ absolute units;}$$

or if m and m' be gram weights, the pull is a stretching force equal to the weight of $\frac{2mm'}{m + m'}$ grams.

§ 87. **Impact.**—The case of two impinging bodies is a good illustration of the application of the third law of motion.

If one body strike another body and change in any way the motion of the other body, its own motion will be changed in an equal quantity and in the opposite direction. Thus if a body *A* strike or impinge on another body *B*, the force which *A* exerts on *B* is equal to the force which *B* exerts, in the opposite direction, on *A*; or the momentum which *A* loses is equal to that which *B* gains; and the total momentum of the two bodies is unchanged. We shall only consider the case of two bodies moving in the same straight line; but the same principles are applicable to any relative direction of motion, and it is only the mathematical calculations that are beyond the scope of the present work.

§ 88. **Perfectly Inelastic Bodies.**—Let the two bodies *A* and *B* be moving in the direction

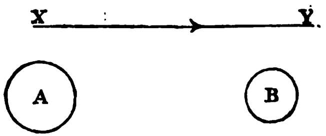

FIG. 25

X Y with such velocities that *A* overtakes *B*. Then, if the bodies are *perfectly inelastic*, i.e. if

the stresses set up in the bodies are too feeble to restore the bodies to their original shape, and can therefore be neglected, the two bodies will move on together with a common velocity, which may be thus determined :—

Let the velocity of A be u and its mass m,
and " " B " u' " " m'.

Then the body A has $m\,u$ units of momentum
and " " B $m'\,u'$ " "

Let v be their common velocity after having impinged ; then, since action and reaction are equal and opposite, the momentum lost by A equals the momentum gained by B, or

$$m\,(u - v) = m'\,(v - u'),$$

or $$(m + m')\,v = m\,u + m'\,u' ;$$

i.e. the total momentum after impact equals the total momentum before impact, whence

$$v = \frac{m\,u + m'\,u'}{m + m'}.$$

If the bodies are moving in opposite directions before impact, then one of the velocities—say u'—must be considered negative, in which case

$$v = \frac{m\,u - m'\,u'}{m + m'}.$$

If, in this case, $m\,u = m'\,u'$, $v = 0$, i.e. if the two bodies before impact are moving in

opposite directions, but so that the momentum of the one equals the momentum of the other, the effect of the collision will be to bring the two bodies to rest.

§ 89. **Examples.**—(1) Two inelastic bodies the masses of which are 5 lbs. and 9 lbs. are moving in the same direction with velocities of 10 and 5 respectively; what is their common velocity after impact?

Here $m = 5$, $m' = 9$, $u = 10$, and $u' = 5$; $\therefore$ total momentum before impact equals $5 \times 10 + 9 \times 5 = 95$, and total momentum after impact $= 14\,v$.

$$\therefore\ v = \frac{95}{14} = 6\tfrac{11}{14}.$$

(2) Two bodies whose masses are 10 lbs. and 8 lbs. are moving in opposite directions with velocities of 4 and 6 respectively; find the velocity and direction of the motion after impact.

Here $v = \dfrac{10 \times 4 - 8 \times 6}{18} = -\tfrac{4}{9}$ foot per second.

The motion is thus in the same direction as that of the mass of 8 lbs.

§ 90. **Elastic Bodies.**—If the impinging bodies are not perfectly inelastic, but are perfectly or imperfectly elastic, i.e. if the internal stresses set up by impact are sufficient to wholly or partially restore the bodies to their original form, the two bodies will rebound or separate from one another and move with different velocities. This

is what happens when two billiard-balls strike one another, and is what usually occurs when one body impinges on another. In this case the motion after impact may be thus determined :—

Let v be the velocity of the body A after impact,
v' „ „ B „

Then the total momentum after impact is $m\,v + m'\,v'$, and from the third law of motion it follows that

$$m\,u + m'\,u' = m\,v + m'\,v',$$

where u and u' are the velocities of A and B *before* impact. Now, by actual experiment, it is found that, for any two materials, the relative velocity, i.e. the rate at which the two bodies *approach* each other, *before* impact always bears a constant ratio to the relative velocity, i.e. the velocity with which they *recede* from each other, *after* impact.

The velocity of approach before impact is $u - u'$, if A overtakes B.

The velocity of separation after impact is $-(v - v')$, and experiment shows that the ratic of these two velocities is constant, or $\dfrac{-(v - v')}{u - u'} = e$, where e is constant between any two materials. If the bodies were perfectly elastic e would equal unity, but for all bodies imperfectly elastic it is

less than one. The number e is called the *coefficient of restitution.*

From the equations $m u + m' u' = m v + m' v'$ and $e(u - u') + v - v' = 0$, v and v' may be found in any given case. If the bodies are perfectly inelastic, as previously supposed, $e = 0$ and $v = v'$; i.e. the bodies move on together with a common velocity.

§ 91. **Impulse.**—The case of impact which we have considered in the foregoing paragraph is very similar to the effect which we have called an *impulse*, and have discussed in a previous section. An impulse is generally an instance of impact in which one of the impinging bodies is restrained. Thus when a bat strikes a ball the effect of the blow is to transfer momentum from the bat to the ball, but whilst the ball moves off with the velocity imparted to it the motion of the bat in the opposite direction is restrained.

§ 92. **Recoil.**—The third law of motion is also well illustrated by the recoil of a gun when a shot is projected from it. This, again, is a case of impulse. Without considering the exact action that takes place in consequence of the expansion of the gases which the ignited powder evolves, the total effect is such that the momentum of the

shot is equal in magnitude and opposite in direction to the momentum of the gun.

§ 93. **Example.**—Suppose a 48-lb. shot to start from a gun of 4 tons mass with a velocity of 200 feet per second; the velocity of recoil can be easily determined; for if v equal the velocity,

the total momentum before explosion = 0

„ „ after „ $= 8960\,v + 48 \times 200$; since 4 tons = 8960 lbs.

$\therefore\ 8960\,v + 9600 = 0$, or $v = -1\frac{1}{14}$ foot per second; e.g. the gun moves in an opposite direction to the shot with a velocity of $1\frac{1}{14}$ foot per second.

It follows as a corollary from the third law of motion that the mutual actions and reactions of the different members of a system of bodies cannot change the total momentum of the system.

EXERCISES IX

1. Two masses of 120 grams and 140 grams are connected by a fine thread which passes over a smooth wheel that revolves without friction; find the acceleration, and the pull in the string. ($g = 980$.)
2. Show how to find the pull, or so-called 'tension,' of the string and the acceleration when one ball is drawn up an inclined plane by another which hangs by a string passing over a fixed pulley at the top of the plane.
3. Apply the third law of motion to determine the velocity gained per second when masses of 6 oz. and 4 oz. are attached to the two ends of a string passing over the edge of a smooth table, the greater being drawn along the table by the smaller, which descends vertically.

4. A rifle is pointed horizontally with its barrel 5 feet above a lake. When discharged the ball is found to strike the water 400 feet off. Find approximately the velocity of the ball.
5. Two bodies, perfectly inelastic, of different masses, are moving towards each other with velocities of 10 and 12 respectively, and continue to move after impact with a velocity of 1·2 in the direction of the greater. Compare their masses.
6. Two bodies of equal masses are moving in the same direction, and the velocity of the one is 10 feet per second, of the other 15 feet per second; find their velocity after impact.
7. Two bodies whose masses are to one another as 3 : 2 are moving with velocities of 20 and 25 respectively (1) in the same direction, (2) in opposite directions; compare their velocities after impact.
8. A body the mass of which is 10 lbs. is projected along a smooth horizontal plane with a velocity of 20 feet per second and strikes another body at rest the mass of which is 40 lbs.; find the velocity with which they move on together.
9. Three bodies of equal masses are placed at equal distances in a straight line on a smooth horizontal plane, a fourth body of equal mass is projected in the same line with a velocity of v feet per second; find the velocities after successive impacts.
10. If a shot of mass 20 lbs. leave a gun of mass 3 tons with a velocity of 1,200 feet per second, find the velocity of the gun's recoil.
11. If a shot weighing 32 lbs. be fired from a gun weighing 2 tons with a velocity of 1,120 feet per second, and if the friction between the gun and the ground be equal to a force of 1 ton, how far will the gun recoil?
12. Equal spherical inelastic bodies are placed at short equal intervals in a smooth horizontal groove. The first is projected from an end along the groove with

a velocity of 20 feet per second. Find the velocities after successive impacts.

13. Determine the velocities after impact of two balls whose masses are 300 grams and 500 grams, moving in the same straight line with velocities of 120 and 150 cents. per sec. respectively, (1) when they are moving in the same direction, (2) when they are moving in opposite directions. ($e=\frac{2}{3}$.)

14. The masses of two balls A and B in the same straight line are 50 grams and 100 grams respectively. Before impact A is at rest, and B is moving towards it with a velocity of 150 cents. per sec. Find the velocities of the balls after the impact. ($e=\frac{1}{2}$.)

15. An elastic ball mass m strikes a hard surface at right angles to its direction with a velocity n; find the velocity of rebound if e be the coefficient of restitution between the ball and the surface.

16. Find the coefficient of restitution if a ball fall on a hard surface from a height of 16 feet and rebound to a height of 9 feet. ($g=32$.)

17. In a large hotel there is a lift-chamber, from the roof of which a mass hangs by means of an india-rubber string. Suddenly the support of the lift-chamber gives way, and the chamber, with all that it contains, falls freely down under the action of gravity. What will now first happen to the mass and its india-rubber support?

EXAMINATION QUESTIONS IV

1. The two ends of a string passing over the pulley of an Atwood's machine are loaded as follows: A with $16\frac{1}{2}$, B with $15\frac{1}{2}$ oz. Find the 'tension' at A when it is in motion downwards.—*Univ. of Lond. Matric.*, June 1877.

2. A half-ton shot is discharged from an eighty-one-ton

gun with a velocity of 1,620 feet per second. What will be the velocity with which the gun will recoil, if the mass of the powder be neglected?—*Ib.* Jan. 1882.

3. Two heavy bodies are connected by a flexible string which passes over a fixed pulley. Show how to find the acceleration with which the heavier body will descend.

   If the masses of the two bodies are respectively 17 oz. and 15 oz., find the 'tension' of the string.—*Ib.* June 1884.

4. Two weights, of 5 lbs. and 7 lbs. respectively, are fastened to the ends of a cord passing over a frictionless pulley supported by a hook. Show that when they are free to move the pull on the hook is equal to $11\frac{2}{3}$ lbs. weight.—*Ib.* Jan. 1886.

5. A particle whose mass is 10, moving with a velocity 5, meets and impinges directly on another particle whose mass is 20 and velocity 3, the coefficient of restitution being 0·125. Find from first principles the velocities of the particles at the end of the impact.—*S. & A. Dep.*

6. A body is constrained to move in a circle by means of a string fastened to the centre; the radius of the circle is 3 feet, the mass of the body 12 lbs., and the velocity 40 feet a second. Find the 'tension' of the string (*a*) in poundals, (*b*) in pounds.—*Ib.* 1887.

CHAPTER V

ENERGY

XII. *Work—Power*

§ 94. **Work.**—Whenever a body moves through any distance in consequence of the action of a force upon it, the force is said to do *work*. If a horse draw a cart along a rough level road, the horse does a certain amount of work, which depends on the force exerted by the horse and on the distance through which the cart is moved. When the body moves in the opposite direction to that in which the force is acting, the force is said to have *work done against it*. Thus suppose a heavy mass to be raised against the action of its weight, work would be done against the weight. In the former case the work is reckoned as *positive*, in the latter case as *negative*. A force which acts in the direction in which its point of application moves is sometimes called an *effort*, and when it acts in the opposite direction to that in which its point of application moves it is called a *resistance*. Thus the force exerted by the horse when drawing

the cart is an *effort*, and the force against which he exerts this effort is the *resistance* due to the friction between the cart and the road.

§ 95. **Units of Work.**—The unit of work is the work done by a unit force acting through a unit distance. This unit depends on the units adopted for force and length. Thus if a *poundal* act through a *foot* the unit of work is a *foot-poundal*, and if a *dyne* act through a *centimetre* the work done is a *dyne-centimetre*, or *erg*, the name given to the unit of work in the C.G.S. system. The *erg* is a very small quantity, and is not much used in actual practice. A more convenient unit is that obtained by considering the work done in raising a pound against gravity through one foot. In this case the measure of the force of gravitation is one pound weight, and the distance moved through is one foot. This unit is called a *foot-pound*, and is the unit commonly adopted by engineers.

In the metric system the corresponding unit is the *kilogram-metre*, which may be defined as the work done against gravity in the ascent (or that done by gravity in the descent) of one *kilogram* through one *metre*.

The foot-pound $= 32{\cdot}2$ foot-poundals, taking $g = 32{\cdot}2$ (feet, seconds) $= 32{\cdot}2$ (f., s.)

The kilogram-metre $= 981 \times 10^5$ ergs, taking $g = 981$ (centimetres, seconds) $= 981$ (c., s.)

The gravitation units of work, like the gravitation units of force, are subject to variation according to the distance of the place from the earth's centre.

It will be seen that whether we fix our attention on the *effort* exerted to raise a pound through a foot, or on the *resistance* offered by gravitation to the motion, the result is the same.

§ 96. **Measure of Work.**—Since the unit of work is the work done by a unit force acting through a unit of distance, if a force F act, in its own direction, through a distance s, the measure of the work done will be Fs, and this is equally the case if F act in the opposite direction to that in which motion takes place. The work done is measured, therefore, by the product of the *effort* or *resistance* and the *effectual distance* through which the body moves. By *effectual distance* is meant the distance measured in the direction in which the force acts. Thus if a man pull a mass of 100 lbs. up a smooth incline 200 feet in length and 80 feet in height, the *effectual distance* through which the gravitational resistance has been overcome is 80 feet, although the distance traversed by the mass is 200 feet. Hence the measure of the work is $80 \times 100\,g$ foot-poundals, or 8,000 foot-pounds. Generally if a force F act on a body at the point A in the direction $A\,T$, and the point of

application A move from A to B, then the *effectual* distance through which A is moved, i.e. the distance in the direction of the action of the force, is $A\,M$, where $B\,M$ is perpendicular to $A\,T$. Hence the measure of the work done is $F \times A\,M$.

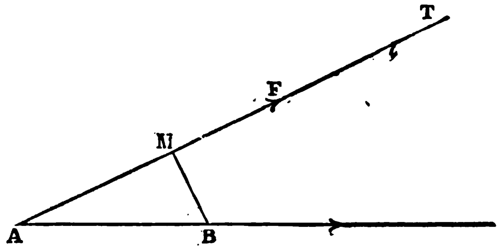

FIG. 26

It follows that a force acting on a body does no work when the body (or more correctly the point in the body at which the force is applied) moves in a direction at right angles to that in which the force acts, for in that case the *effectual* distance $A\,M = 0$.

§ 97. **Power.**—The power of an agent is measured by the quantity of work it can perform in a given time. Thus if one labourer can do more work in an hour than another, the former is said to be more powerful. Power is thus the *rate of doing work*, and may be measured by the work done in one second.

The *unit* of power, accordingly, is the power

exerted by a force that does one unit of work in a second. This unit may be expressed as an erg per second or a foot-poundal per second; or, if we take gravitation units, as a kilogram-metre or a foot-pound per second.

James Watt came to the conclusion that the power of a good horse was 33,000 foot-pounds per minute, or 550 foot-pounds per second; and since his time it has been usual, especially among engineers, to call this rate of working one *horse-power*, denoted by the symbol HP. This unit is found very convenient for expressing the power of steam engines, gas engines, and prime movers generally. It is equivalent to $7{\cdot}46 \times 10^9$ ergs per second $(g = 981)$.

If we call P the power of an agent which does E units of work in t seconds, $P = \frac{E}{t}$, or $E = P.t$.

§ 98. **Examples.**—(1) What is the horse-power of an engine that can raise every minute and a half 2 tons of water to a height of 100 feet?

The work done in $1\frac{1}{2}$ minute is $2 \times 20 \times 112 \times 100$ foot-pounds; and since one horse-power does 33,000 foot-pounds in a minute, the required horse-power is

$$\frac{2 \times 20 \times 112 \times 100}{\frac{3}{2} \times 33000} = 9 \text{ horse-power nearly.}$$

The rate of working $= \dfrac{2 \times 20 \times 112 \times 100}{\frac{3}{2} \times 60} = 4977{\cdot}7$ foot-pounds per second.

(2) How much work is done when an engine weighing 10 tons moves half a mile on a horizontal road, if the total resistance is equal to a retarding force of 8 lbs. weight per ton?

The resistance is equal to 80 lbs. weight, and distance traversed is 2,640 feet;

$$\therefore \text{ work done } = 80 \times 2640 \text{ foot-pounds.}$$

(3) If a mass weighing a half-ton be raised by 20 men to a vertical height of 20 feet twice in a minute, how much work does each man do per hour; and what is the equivalent horse-power of the gang?

Here the resistance due to the weight of the mass is 1,120 lbs. weight, and the distance moved through is 40 feet per minute;

$\therefore$ the power exerted, or work done per minute, is $1120 \times 40 = 44{,}800$ foot-pounds; and the work done per hour is $60 \times 1120 \times 40 = 2{,}688{,}000$ foot-pounds;

$\therefore$ each man does 134,400 foot-pounds per hour,

and the equivalent horse-power of the gang is $\frac{44800}{33000} = 1\frac{59}{165}$ horse-power.

(4) If a train whose mass is 40,000 kilograms, moving on a level road at the rate of 3,000 metres per hour, is brought to rest, find the work done against the resistance, if the total resistance is equal to $\frac{1}{200}$ of the weight of the train.

The weight of the train is $4 \times 10^7 \, . \, g$ dynes,

and $\therefore$ the total resistance is $2 \times 10^5 \, . \, g$ dynes;

$$\therefore \text{ the retardation } a = \frac{2 \times 10^5 \, . \, g}{4 \times 10^7} = \frac{g}{200} \text{ (c., s.),}$$

and $v = 3000$ m. per hr. $= \dfrac{300000}{60 \times 60} = \dfrac{10^3}{12}$ cents. per sec.;

and since $v^2 = 2\,a\,s$, where s is the distance traversed,

we have $\dfrac{10^6}{144} = \dfrac{2\,g}{200}.s$, or $s = \dfrac{10^8}{144\,g}$ centimetres;

∴ the work done against the resistance equals

$$2 \times 10^5 . g \times \frac{10^8}{144\,g} \text{ ergs } = \frac{10^{13}}{72} \text{ ergs.}$$

The kilogram-metre $= 981 \times 10^5$ ergs;

∴ the work done is $\dfrac{10^8}{981 \times 72} = 1415$ kilogram-metres nearly.

EXERCISES X

1. How much work is done per hour if a mass of 100 lbs. be raised 3 feet in 1 minute?
2. A man weighing 140 lbs. carries a load of 100 lbs. up a ladder to a height of 50 feet; how many foot-pounds of work does he do altogether, and what part of his work is done usefully?
3. If a pit 10 feet deep and with an area of 4 square feet be excavated, and the earth thrown up, how much work will have been done, supposing a cubic foot of earth to weigh 90 lbs.?
4. What power must be exerted by an engine to draw a train weighing 55 tons along level rails at a uniform speed of 30 miles per hour, the total resistance being equal to $\frac{1}{100}$ of the weight of the train?
5. If a man does 1,056,000 foot-pounds of work in a day of 8 hours, at what fraction of a HP does he work on the average?
6. What is the HP of a waterfall of 18 feet height when the stream above the fall passes through a section of 6 square feet at the rate $2\frac{1}{2}$ miles an hour?

7. A body weighing 8 cwt. is drawn along 100 feet up a smooth incline which rises 2 feet in height for every 5 feet along the incline; how much work is done?
8. How many foot-pounds of work are required to raise 30,000 lbs. of water from the depth of a furlong, and how many HP's to do it in five minutes?
9. A train weighing 10 tons moves for 20 minutes at a uniform rate of 30 miles an hour; if the friction, &c., be 10 lbs. per ton, how much work is done during the time?
10. Compare the *erg* with a foot-pound where $g=981$ (c., s.), having given 1 foot $=30{\cdot}48$ centimetres and 1 lb. $=453{\cdot}6$ grams.

XIII. *Friction*

§ 99. **Friction.**—The resistance which a force has to overcome in moving a body may be due to various causes, one of which, viz. gravitation, has already been considered. Another, quite as common, is friction, which is brought into play whenever one rough body is made to slide over another. Practically all bodies are rough; for although there exists hardly any limit to the degree of polish that can be imparted to certain substances, still no surface can ever be absolutely smooth.

§ 100. **Measure of Friction.**—It is evident that the resistance of friction acts in the opposite direction to that in which the body tends to move. It can be measured, therefore, by the force that is necessary to move the body on a horizontal plane,

since in this case no part of the force used is employed in raising the body against its weight. Suppose we require to know the resistance of friction that exists between a body of weight W and the surface $A\ B$ on which it rests. Let a fine thread be attached to the body and passed over a small wheel, as in fig. 27, and let the other end of the thread support a mass of weight P. If P be the least weight which will cause the body to slide

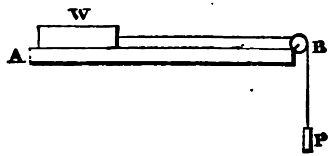

Fig. 27

along the plane, P is the measure of the friction between the two surfaces

§ 101. Another method of measuring the resistance due to friction is by elevating the plane $A\ B$ till the body begins to slide. If $A\ B\ C$ be the angle of elevation of the plane at which sliding commences, the resistance of friction is equal to that force which, acting parallel to the plane, would support the weight W, the plane being supposed perfectly smooth. If F be this force $F : W :: h : l$, where h and l are the height and length of the plane (§ 72, Ex. 13); $\therefore F = \frac{h}{l} W$.

If R be the normal or statical reaction of the surface it can be shown that $R : W :: b : l$ where $b = B\,C$ the base of the plane, and hence

$$F = \frac{h}{l} \cdot \frac{l}{b} R = \frac{h}{b} R.$$

The angle $A\,B\,C$ is called the *angle of friction.*

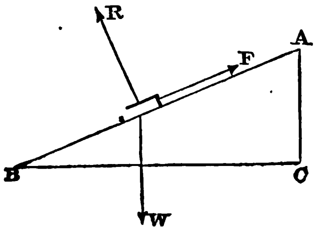

FIG. 28

§ 102. **Laws of Friction.**—By these means the friction between the surfaces of different substances has been determined, and the following laws have been established as the results of experiments by Coulomb and Morin :—

1. The friction depends on the nature of the surfaces, but is independent of the extent of the areas in contact.

2. The friction varies with the normal reaction when the materials of the surfaces in contact remain the same.

3. The friction is independent of the velocity when there is sliding motion.

These laws must not, however, be regarded as rigorously true under all circumstances. As regards Law 3, subsequent experiments have shown that the friction diminishes as the velocity increases beyond a certain limit, and that the force necessary to overcome friction when sliding is about to take place is greater than when one body is actually sliding on another. In the former case the friction is called *static* and in the latter *kinetic* friction.

§ 103. **Coefficient of Friction.**—The ratio of the friction to the normal reaction, when sliding is about to take place, is called the *coefficient of friction*, and is represented by the Greek letter μ. Thus, if F be the force parallel to the surfaces in contact, which is just sufficient to produce sliding motion, and if R be the normal reaction, $\frac{F}{R} = \mu$, or $F = \mu R$.

If a heavy body rest on a horizontal surface the normal reaction equals the weight, or $F = \mu W$; but if it rest on an inclined plane, as in fig. 28, $F = \mu R = \mu \frac{b}{l} W$.

It follows from § 101 that the coefficient of friction is the ratio of the height to the base of the plane on which sliding is about to commence.

§ 104. **Examples.**—(1) If the smallest force which will move a body of 10 lbs. weight along a rough hori-

zontal plane be equal to the weight of $5\sqrt{2}$ lbs., find the angle of friction.

Since $F = \mu W$, we have $5\sqrt{2} = \mu\, 10$, or $\mu = \dfrac{1}{\sqrt{2}}$.

The plane may be inclined through an angle of 45° before the body will commence to slip down the plane.

(2) A body is projected along a rough horizontal plane with a velocity of 100 feet per second; if the coefficient of friction between the body and the plane is $\frac{1}{5}$, how far will it move before coming to rest?

Let m be the mass of the body, then $m\,g$ is its weight,

$$\text{and since } F = \mu W,\ F = \tfrac{1}{5}\, m\, g;$$

$$\therefore \text{ retardation } a \text{ due to friction} = \frac{\frac{1}{5}\, m\, g}{m} = \frac{g}{5}$$

$$\text{and since } u^2 = 2\, a\, s, \text{ we have } (100)^2 = \frac{2\,g}{5} \cdot s,$$

$$\text{or } s = 781\tfrac{1}{4} \text{ feet, taking } g = 32.$$

(3) Find the work done in drawing a body of weight W up a rough inclined plane the height of which is h and length l.

Let μ be the coefficient of friction. The work done against gravity is $W\,h$, since the body of weight W is lifted through height h. The work done against friction is $F\,l$, where F is the resistance parallel to the plane due to friction. Now, we have seen that $F = \mu\,\dfrac{b}{l}\,W$, where b is the base of the plane. Therefore the work done against friction is $\mu\, b\, W$; and the total work done equals $W\,h + \mu\, b\, W$, or the same amount as would have been done if the body had been moved along the base of the plane horizontally against friction, and then elevated against gravity through a height h.

EXERCISES XI

1. Find the work done in sliding a cast-iron body of 56 lbs. weight along a horizontal plane of 15 feet long, the coefficient of friction being 0·55.
2. A mass of 500 lbs. by falling through 36 feet lifts by means of a machine a mass of 60 lbs. to a height of 200 feet. How many units of work have been expended on friction, and what proportion does the expenditure bear to the whole amount of work done?
3. A heavy body is just on the point of sliding on a rough plane that rises 3 in a length of 5; find the coefficient of friction.
4. What is the pull of a string which, being stretched, is just able to move a body weighing 10 oz. over a rough horizontal plane, coefficient of friction being $\frac{2}{5}$?
5. A rough plane rises 3 in 10, and a body weighing 12 oz. is just supported by friction; find the force of friction.
6. Find the work expended in pulling a body weighing 3 cwt. 100 yards up an incline that rises 1 in 10, if the force of friction be 10 lbs. per cwt.
7. An inclined plane has a base 120 feet long, and is 50 feet high; the coefficient of friction between it and a body weighing 56 lbs. placed on it is 0·5: how many units of work are required to draw the body up the plane, and how many to draw it down the plane?
8. A body weighing 12 oz. is partly supported on a rough inclined plane that rises 1 in 2 by a force of friction and partly by a force parallel to the plane; if the plane be lowered so as to rise 2 in 5, the force of friction alone will support it; find the additional force in the former case.

XIV. *Varieties of Energy—Relation between Units*

§ 105. **Definition of Energy.**—The *energy* of a body is its capacity for doing work. Every moving body possesses energy, i.e. it is capable of overcoming resistance. Instances of bodies possessing energy are numberless. The flying bullet can pierce a sheet of iron by overcoming the cohesion between its particles ; the running stream is able to turn the wheel of the water-mill, and the energy it possesses may be utilised in grinding corn ; the moving air drives the ship through the water and overcomes the resistance offered to its passage. Wherever we find matter in motion, be it solid, liquid, or gaseous, we have a certain amount of energy which can be utilised in endless ways. There are other forms of energy besides matter in motion, some of which we shall here consider.

§ 106. **Measure of Kinetic Energy.**—The energy of moving matter is called *Kinetic Energy*, or *Energy of Motion* ; it is energy ready for use—energy that is generally being spent, though it may not be economically employed. The rivulet will flow from the highlands to the lowlands, whether we give it work to do or not. We know

all about the motion of a body if we know its velocity and mass; we can, therefore, express its energy in terms of these two quantities. Let m be the mass of the body, and v its velocity, and suppose the force F acting on the mass m through the distance s reduces the velocity v to zero. Then, if a equals the retardation, $F = m\,a$. Also $v^2 = 2\,a\,s$, $\therefore F = \frac{m\,v^2}{2\,s}$, or $F\,s = \frac{m\,v^2}{2}$. But $F\,s$ equals the total work done against the resistance F, $\therefore \frac{m\,v^2}{2}$ equals the total number of units of work in the mass m moving with a velocity v. The quantity $\frac{m\,v^2}{2}$ is therefore the measure of the kinetic energy of the mass m. It will be seen that this energy is expressed in units of work, and the unit of energy may be defined as the energy of a moving mass which is capable of doing a unit of work, or of transferring a unit of work to some other body in coming to rest.

In gravitation units the kinetic energy of a mass m moving with a velocity $v = \frac{m\,v^2}{2\,g}$, the units being *foot-pounds*, or *gram-centimetres*, or *kilo-gram-metres*, according to the system of units adopted.

If the force F' change the velocity of the mass

m from v to u in passing over a distance s, we have

$$F s = \frac{m v^2}{2} - \frac{m u^2}{2};$$

or the work done by a mass m, moving against a resistance which changes its velocity from v to u, is $\frac{1}{2} m (v^2 - u^2)$.

§ 107. **Potential Energy.**—Matter is not deprived of energy even when at rest. We have seen that, because of the mutual forces or *stresses* existing between all bodies in the universe, and also between the particles of bodies, bodies at rest have a tendency to motion, which the removal of a force, or the application of a force, can always render actual; and the moving body, in virtue of this motion, will be capable of doing work. Now this capability of doing work must have existed in the apparently inert body before the opposing force was removed or the new force applied; and the body, therefore, must have possessed a store of energy due to its position relatively to other bodies (as, for example, the earth) or to the relative position of its particles, and this energy is called *Potential Energy*, or *Energy of Position*. Suppose a mass weighing 1 lb. be projected vertically upwards with a velocity of 32·2 feet per second. The energy imparted to the body will have carried it to a height of 16·1 feet, and the body will then

cease to have any velocity. The whole of its kinetic energy $\frac{m v^2}{2}$ will have been expended in raising the mass m to a height of 16·1 feet against the resistance $m g$, i.e. $\frac{m (32{\cdot}2)^2}{2} = m \times 32{\cdot}2 \times 16{\cdot}1$; but the body will have acquired, instead, a new position, a vantage-ground; and, if free to fall from this position it will obtain in reaching the point of projection a velocity of 32·2 feet per second, and thus re-acquire the energy which it originally received. Now, we may suppose the body to be lodged for any length of time at an elevation of 16·1 feet above its point of projection, and during this time its energy will be *potential* —stored up and ready to be freed whenever it shall be permitted to fall. The energy which in bygone years has been expended in raising walls and towers on the tops of hills and elsewhere still survives, and when the stones of which they are composed shall fall from their places there will be expended in the fall the same amount of energy as was employed in raising them. In the boulder embedded in the sea-shore we have evidence of kinetic energy that has been spent; in the overhanging crag we see a mass endowed with the energy of position, which at any moment, by the loosening of forces, may be changed into destructive work. Nature by her own processes is continually

modifying the relative position of the matter of which this earth is formed, and in every change there is a readjustment of energy, but neither gain nor loss. Energy may exist in many forms, but they all fall under one or other of the two classes already named—energy of motion or of position.

§ 108. **Conversion of Heat into Mechanical Work.**—Of the various forms of energy none is more universal than that of heat. Our chief sources of heat are the sun and the combustion of fuel. The sun's heat is being constantly changed into mechanical energy. The wind that turns the sails of the mill and drives the ship through the sea receives its motive power from the sun's heat. The energy of a running stream has been derived from the same source. The clouds that settle on the ridges of the hills and on the mountain-peaks have been raised by the energy of the sun's heat from the waters of the sea; and the falling rain feeds the streams and rivulets which, after having expended some part of their energy, find their way back to the sea. The havoc which so often follows the avalanche in its fall is caused by the release of the sun's energy stored up in the snow and ice.

When wood or coal is burned, the heat evolved can be made to perform work. The steam engine is the most important instrument for converting heat-energy into mechanical energy. A portion of

this mechanical energy is employed in overcoming certain resistances incidental to the working of the machine; the remainder may be spent directly in work, as in raising heavy loads by the steam crane, or in giving motion to other bodies, as in the ordinary locomotive. In either case the heat is transmuted into work.

§ 109. **Conversion of Mechanical Energy into Heat.**—In discussing Newton's third law of motion we showed that when one inelastic body impinges on another, the momentum lost by one body is gained by the other, and that if they are moving in the same direction the momentum of the two bodies is the same before and after impact, having regard to the algebraic signs of the velocities involved. But although the momentum is unchanged by the impact, it is not so with the kinetic energy. If we suppose two bodies, the masses of which are 4 lbs. and 10 lbs., to be moving in the same direction, with velocities of 15 and 8 feet per second respectively, their common velocity after impact will be 10 feet per second (§ 88). Now the kinetic energy before impact $= \frac{1}{2}(4 \times 15^2 + 10 \times 8^2) = 770$, and the kinetic energy after impact $= \frac{1}{2}(10 + 4) \times 10^2 = 700$. Thus there is a loss of visible kinetic energy. If the two bodies are moving in opposite directions, or if an inelastic body in motion strike a similar body at rest, there will be a still greater

loss of kinetic energy. In all these cases, however, heat is generated by the impact, and careful experimental investigation has shown that the heat generated is proportional to the kinetic energy lost.

We have seen that when one body moves on the surface of another body a resistance is offered to the motion, which is due to friction. If a body be projected along a rough road with a certain velocity, it will after a time come to rest, and the kinetic energy of the body will have been destroyed by friction. Now in this case a corresponding amount of heat is found to have been produced, exactly equivalent to the energy apparently lost. One of the earliest ways of obtaining heat was by rubbing sticks together—in other words, by evolving it from mechanical energy. We have seen that in a locomotive engine heat-energy is changed into kinetic energy; but the heat employed must be in excess of the kinetic energy required, as a large amount of friction has to be ovecome, and this friction reproduces heat. Hence part of the heat-energy evolved from the coals, after having been transformed into mechanical energy, reappears in the increased temperature of the rails, axles, and other parts of the machinery. If the speed of the engine remain uniform, the heat of the furnace is wholly employed in overcoming these resistances, and is spread over the rails and machinery; and

if the steam be turned off and brakes applied, the kinetic energy of the engine is gradually converted into heat, and the locomotive stops. Whenever a body is brought to rest by moving over a rough surface, by passing through water or air, or by striking against another body, the energy of the moving body is not destroyed, but is converted into a new form of energy, equal in amount to that which is apparently lost.

§ 110. **Relation between Units of Work and Heat.**—Careful experiments have shown the exact amount of heat that is equivalent to a unit of work. The details of these experiments are given in treatises on Physics, to which the student is referred for full information respecting other forms of energy. By these experiments it has been proved that the quantity of heat necessary to raise the temperature of a pound of water through 1° F. possesses the same amount of energy as is required to lift 772 lbs. against gravitation through one foot. In other words, the energy of this unit of heat is equivalent to 772 foot-pounds. If 1° C. be taken instead of 1° F., the energy would be 1,390 foot-pounds. In the C.G.S. system, the unit of heat is the *calorie*, and is the quantity of heat required to raise the temperature of one gram of water from 4° C. (its temperature of maximum density) to 5° C. The mechanical equivalent

of this unit of heat, which is usually denoted by the letter J, after Joule, to whose experiments these results are largely due, is about 42,400 gram-centimetres, or 42×10^6 *ergs*. In other words, a *calorie* can generate 42,000,000 *ergs*, and in the destruction of 42,000,000 (C.G.S.) units of mechanical energy one calorie is produced.

§ 111. **Relaticn between Force, Momentum, and Energy.**—From what has gone before, it appears that a body can only be brought to rest by the action of a force or resistance in a direction opposite to its motion. In considering the action of the force, we may have regard to the *time* during which the force acts, or to the *distance* through which it acts. In the former case it may be measured by the change of momentum in a unit of time, in the latter case by the change of energy in a unit of distance. Thus, if a force F, acting for t seconds on a mass M, gives it a velocity of v, we know that $F = Ma$ and $v = at$, so that $F = \frac{Mv}{t}$ or $Ft = Mv$. And if a force F, acting through a distance s on a mass M, gives it a velocity v, we know that $Fs = \frac{Mv^2}{2}$, or $F = \frac{Mv^2}{2s}$. The two expressions $F = \frac{Mv}{t}$ and $F = \frac{Mv^2}{2s}$ show us that the force F may be considered as the change of *momentum* with respect to *time*, or as the change

of *energy* with respect to *distance*; and if we make F, t, and s each equal unity, we see that the unit of *momentum* is produced by a unit of force, acting for a unit of *time*, and that the unit of *energy* is produced by a unit of force acting through a unit of length.

It follows from the above, that if two masses have the same momentum, that which has the greater velocity, and consequently the less mass, has the greater kinetic energy. For if $Mv = M'v'$ and v is greater than v', $\frac{v}{2}$ will be greater than $\frac{v'}{2}$, and therefore $\frac{Mv^2}{2}$ is greater than $\frac{M'v'^2}{2}$. In discussing the third law of motion we saw that in firing a gun the momentum of the shot forward is equal to that of the gun backward; but the kinetic energy of the shot, i.e. the work it is capable of doing against a resistance, is greater than the kinetic energy of the recoiling gun.

§ 112. **Examples.**—(1) A body is projected up a rough inclined plane which rises 1 in 10 with a velocity of 40 feet per second. If the resistance due to friction is $\frac{1}{100}$ the weight of the body, find how far the body will move up the plane.

Here the kinetic energy of the body on starting is $\frac{1}{2}Mv^2 = \frac{1}{2}M \times (40)^2 = 800\,M$; and this energy is employed in overcoming the resistance due to friction and to gravitation.

The resistance due to friction is $\frac{Mg}{100}$;

$\therefore$ the work done against friction is $\frac{Mg}{100} \cdot s$, where s is the distance moved through up the plane.

The work done against gravity is $\frac{Mg}{10} \cdot s$;

$$\therefore 800\,M = \frac{Mg}{10} \cdot s + \frac{Mg}{100} \cdot s;$$

$\therefore s = \frac{80000}{11 \cdot g} = 227\frac{3}{11}$ feet, taking $g = 32$ (f., s.)

(2) A bullet of mass 1 oz. leaves the muzzle of a gun 3 feet in length with a velocity of 1,000 feet per second; find the average force of the powder on the bullet.

Here the force acts through 3 feet, and in doing so gives to the bullet a velocity of 1000.

The work done by the explosive force of the powder is $3\,F$ foot-poundals, and the energy imparted to the bullet is $\frac{(1000)^2}{2} \times \frac{1}{16}$ foot-poundals; or $3\,F = \frac{10^6}{32}$,

$$\therefore\ F = \frac{10^6}{96} \text{ poundals, or } \frac{10^6}{96\,g} \text{ pounds weight.}$$

(3) A heavy body slides down the whole length of a rough inclined plane, and with the energy acquired slides up another rough inclined plane in contact with it; find the distance through which it ascends the second plane.

Let h, b, and μ be the height, base, and coefficient of friction on one plane. Let h', b', and μ' be the height, base, and coefficient of friction on the other. Then if W be the weight of the body, $Wh - \mu\,Wb$ is the number of units of work acquired in falling the

whole length of the first plane. Let x be the distance measured horizontally, and y the distance measured vertically, from the foot of the plane, of the point which the body is enabled to reach on the incline.

Then, $Wy + \mu' Wx$ is the measure of the work expended in reaching this point, the positive sign

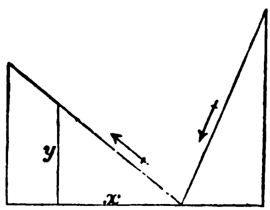

FIG. 29

showing that both gravitation and friction are opposed to the rising of the body.

$$\therefore\ Wy + \mu' Wx = Wh - \mu Wb,$$

or
$$y + \mu' x = h - \mu b;$$

also
$$\frac{y}{x} = \frac{h'}{b'}\ \therefore\ x\left(\frac{h'}{b'} + \mu'\right) = h - \mu b,$$

or
$$x = \frac{h - \mu b}{h' + \mu' b'} \cdot b'$$

and
$$y = \frac{h - \mu b}{h' + \mu' b'} \cdot h';$$

$$\therefore \text{ distance required} = \sqrt{(x^2 + y^2)} = \frac{h - \mu b}{h' + \mu' b'} \times l',$$

where l' is the length of the second plane.

EXERCISES XII

1. What is the measure of the force which acting on a mass of 50 grams reduces its velocity from 100 centimetres per second to 20 centimetres per second, whilst the body traverses $2\frac{1}{2}$ metres?
2. Through what distance must a force equal to the weight of $\frac{1}{2}$ lb. act upon a mass of 48·3 lbs. in order to increase the velocity from 24 feet to 36 feet per second?
3. A body weighing 40 lbs. is projected along a rough horizontal plane with a velocity of 150 feet per second; the coefficient of friction is $\frac{1}{8}$: find the work done against friction in five seconds.
4. What amount of energy is acquired by a body weighing 30 lbs. that falls through the whole length of a rough inclined plane, the height of which is 30 feet and the base 100 feet, the coefficient of friction being $\frac{1}{5}$?
5. A body weighing 20 lbs. is projected down a rough inclined plane (ratio of *height* to *length* = 3 : 5) with a velocity of 30 feet per second. If the coefficient of friction between the surfaces of the body and the plane is 0·2, find the velocity of the body when it has traversed a distance along the plane of 200 feet.
6. A body is projected up a rough inclined plane that rises 7 to a base of 10 with a velocity of 200 feet per second; find its velocity when it returns to the point whence it was projected, the coefficient of friction being 0·3.
7. If $g = 9{\cdot}8$ (metres, seconds), find the equivalent of heat necessary to project 50 kilograms with a velocity of 490 metres per second.
8. A body weighing 50 lbs. is projected along a rough horizontal plane with a velocity of 40 yards per second; what amount of work is expended when the body comes to rest, and what is the equivalent of heat generated?

XV. *Conservation of Energy*

§ 113. **Statement of Law.**—The previous lessons contain a few instances of a vast number of facts, which have been generalised into a fundamental law of physical science. This law, known as that of the Conservation of Energy, embraces the following propositions :—

(1) The sum-total of energy in the universe remains the same.

(2) The various forms of energy may be converted the one into the other.

(3) No energy is ever lost.

§ 114. (1) **Totality of Energy.**—These three propositions are intimately connected with one another. To understand the first of them, we must suppose the universe to have been originally endowed with certain energies, the sum-total of which is always, in quantity, the same. We have seen that various kinds of forces exist. The most important of these is Gravitation, with which matter in all its forms is universally endowed. It acts between bodies in mass, and between the molecules of bodies. Other mechanical forces are Cohesion, which holds the particles of a body together, and Elasticity, which is ex-

hibited in a watch-spring, and which causes the particles of a body to resume their original position after having undergone displacement. We have spoken of the energy of Heat, and have shown the relation between mechanical and thermal energy. The force of Chemical Affinity, which causes the elements which constitute a complex molecule to combine, is not so easily brought within the range of mechanical principles. But whenever this force is allowed to do work, by causing the separated elements to combine (as in the combustion of fuel), the whole of the potential energy lost can be made to appear as heat, and in this way we are able, in most cases, to express, in units of work, the energy due to the existence of this force. Other forces are those which give rise to the phenomena of Light, Electricity, and Magnetism ; but it is beyond the scope of the present work to do more than indicate the existence of such forces. What the law of conservation asserts is, that the sum-total of energy, kinetic and potential, due to all these forces always has remained, and always will remain, the same. If the energy due to any one force be diminished in quantity there must be a corresponding increase in some other variety of energy. It will be seen that this law of the totality of physical energy necessarily involves the second of the three propositions above stated.

§ 115. (2) **Transmutation of Energy.**—Facts have already been mentioned that illustrate this law. The most easily recognisable form of energy is what has been called 'visible energy,' i.e. matter in motion. This can easily be converted into potential energy or energy of position. These two states are exemplified in the oscillation of a pendulum. When the bob of a pendulum reaches its lowest point it possesses a certain amount of kinetic energy, which is sufficient to raise it through a certain arc on the other side, and when it reaches its highest point it has lost all this kinetic energy, and has acquired an equivalent amount of potential energy, which being set free is reconverted into energy of motion. This transmutation of kinetic into potential energy, and of potential into kinetic energy, might go on for ever, if the friction of the pivot and the resistance of the air did not gradually retard the motion of the pendulum. A certain amount of energy thus disappears, and we have found that the energy of motion that is lost is really converted into heat. The pivot and the air become heated by the motion. Kinetic energy, though it may endure for any limited period of time, must ultimately waste away into heat; and all moving bodies, in so far as they move against friction, or through a resisting medium, will eventually come to rest. Perpetual motion, in the sense in which it is

generally understood, is thus demonstrated to be impossible. Since, however, heat is shown to be accompanied by molecular motion, it would appear that the motion of masses is perpetually changing into the motion of molecules.

The connection between heat and motion is better understood than the relation between other forms of energy; but the researches of modern science are continually showing us how other varieties of energy are capable of being transmuted and reproduced under new forms.

§ 116. (3) **The Indestructibility of Energy.**—The law asserts that energy is never lost. It may become latent, and remain so for centuries; but it is permanent. The great source of energy in the Solar System is the sun. Streams of energy are continually flowing from the sun to the earth, and this energy is made to do all kinds of useful work. It gives to the waters of our seas an energy of position, by converting them into vapour and raising them to the hill-tops. It supplies the vegetable world with the energy necessary to separate carbon from the carbonic acid in the air, and the carbon thus fixed in the plant is gradually converted into wood, which may be immediately employed as fuel, or may re-appear as coal after having been buried in the earth for thousands of years. In either case the sun's energy, stored up

in the fuel and in the oxygen of the air, as energy of chemical separation, is eventually set free to be converted into work. The vegetable products of the earth supply food to animals, and these again supply food to man; and thus the energy of animal agents is ultimately derived from the sun. Our scientific knowledge may be too limited to enable us, in all cases, to trace the course of energy through its various transmutations; but every new observation that is made brings with it additional evidence tending to verify the law that energy is never lost.

§ 117. **Dissipation of Energy.**—Although the sum total of the energy of the universe remains the same, the amount of energy that can be employed in doing useful work is found to be continually growing less. Heat is the most generally diffused form of energy; and a study of the laws of heat shows us that in order that heat may be converted into mechanical energy, it is necessary that it should pass from a body of higher to one of lower temperature. This occurs in every heat-engine in the process of converting part of the heat-energy of the steam, gas, or other heated substance into mechanical work. The amount of work that can be got from a given quantity of the substance which is heated largely depends upon the temperature to which it can be cooled, i.e. upon the

difference of temperature at which it is received and ejected. In the fall of temperature, as in the fall of water from a higher to a lower level, work is done, and a certain amount of energy is utilised. On the other hand, we have seen that whenever a body moves against friction a certain amount of heat is generated, as when a brass button is rubbed on a piece of dry wood; and nearly all the mechanical work done by every heat-engine is ultimately resolved into heat-energy, due to friction. But the heat thus generated is at such a low temperature as not to be available for the purposes of work. By no process yet known can this diffused heat be reconverted into mechanical energy. In other ways, too, the temperature of hot bodies is being continually lowered. It thus appears that a large amount of energy, though not absolutely lost, is always becoming practically useless, through the conversion of heat at a high into heat at a low temperature. This loss of available energy, which is known as the 'Dissipation of Energy,' takes place whenever work is obtained from fuel and other sources of heat at a high temperature, and also whenever heat flows by conduction or radiation; and, in the absence of all positive knowledge with respect to the means by which the sun's heat may be renewed, it is thought that 'the mechanical energy of the universe will be more and more transformed

into universally diffused heat, until the universe will be no longer a fit abode for living beings.'

EXAMINATION QUESTIONS V

1. A body, whose mass is 100 grams, is thrown vertically upwards with a velocity of 980 centimetres per second. What is the energy of the body, (1) at the moment of propulsion, (2) after half a second, (3) after one second ?—*Univ. of Lond. Matric.*, Jan. 1878.
2. What is the 'Kinetic Energy' of a moving mechanical system ?

   A shot of 1,000 lbs., moving at 1,600 feet per second, strikes a fixed target. How far will the shot penetrate the target exerting upon it an average pressure equal to the weight of 12,000 tons ?—*Ib.* June 1880.
3. Equal forces act for the same time upon unequal masses, M and m. What is the relation between (1) the momenta generated by the forces, (2) the amounts of work done by them ?—*Ib.* Jan. 1884.
4. What is the relation between the mass and velocity of a cannon-shot, and the work it can do on a fixed target ?

   What is the horse-power of an engine which can project 10,000 lbs. of water per minute with a velocity of 80 feet per second, 20 per cent. of the whole work done being wasted by friction, &c. ?

   (N.B. An agent of 1 horse-power can do 33,000 foot-pounds of work per minute.)—*Ib.* Jan. 1884.
5. A cannon-ball whose mass is 60 lbs. falls through a vertical height of 400 feet: what is its energy ? With what velocity must such a cannon-ball be projected from a cannon to have initially an equal energy ?—*Ib.* June 1884.
6. A body whose mass is 12 lbs. moves from rest with a

uniform acceleration of 100 inches per second; calculate the velocity, momentum, and energy after it has moved over 20 feet. In what units are your results expressed?—*Ib.* Jan. 1890.

7. A body weighing 187 lbs. is supported on an inclined plane, whose angle is 30°, by a horizontal force; find the force and the work necessary to move the body 20 feet along the plane.—*Ib.* Jan. 1890.

8. Distinguish between the momentum and energy of a moving body. A 30-ton mass is moving on smooth level rails at 20 miles an hour; what steady force can stop it (*a*) in half a minute, (*b*) in half a mile? Specify the force completely.—*Ib.* Jan. 1892.

9. A body weighing 10 lbs. is placed on a horizontal plane, and is made to slide over a distance of 50 feet by a force of 4 lbs.; what number of units of work is done by the force? If the coefficient of friction between the body and the plane is 0·3, what number of units of work is done against friction? At the instant the 50 feet have been described, what is there in the state of the body to show that the former exceeds the latter?—*S. & A. Dep.*

10. What is the horse-power of the engine which draws a train at a uniform rate of 45 miles an hour against a resistance of 900 lbs.?—*Ib.* 1887.

11. A particle weighs 10 lbs. and moves at the rate of 1,250 feet per second; find the distance through which it would overcome a resistance of 1,000,000 lbs.—*Ib.* 1888.

12. A number of men can each do on the average 490,000 foot-pounds of work per day of 8 hours; how many of such men are required to do work at the rate of 10 horse-power?—*Ib.* 1889.

13. Find the number of foot-pounds of work required to wind up a given chain which hangs by one end.—*Ib.* 1889.

14. A body whose mass is 10 lbs. is capable of doing 605 foot-poundals of work in virtue of its mass and

velocity; at the rate of how many feet per second is it moving?—*Ib.* 1890.

15. A body impinges directly with a given velocity against a fixed plane. If the mass is 10 lbs. the velocity 20 feet a second, and the coefficient of restitution 0·5, how many foot-poundals of energy disappear in the collision?—*Ib.* 1890.

16. A body whose mass is 6 lbs. is moving at the rate of 8 feet a second; how many foot-poundals of work can it do against a resistance in virtue of its mass and velocity? If it did 117 foot-poundals of work against a resistance, what would thus be its velocity? —*Ib.* 1891.

17. Find the resistance due to friction, if a mass of 10 lbs. ($g=32$), moving with a velocity of 48 feet per second along a rough horizontal plane, passes over 144 feet before coming to rest.—*Royal College of Surgeons.*

CHAPTER VI

MACHINES

XVI. *Application of the Law of Energy—The Lever.*

§ 118. A MACHINE is an instrument by means of which a force applied at one point is able to exert, at some other point, a force differing in direction and intensity. It has been usual in treating of simple machines to call the applied force the 'power;' but the word 'power' is now in general use with an entirely different meaning, as explained in § 97, and we shall accordingly speak of this applied force as the *effort*. The force exerted, or effective resistance overcome, is usually called *the weight*; this resistance may be the earth's attraction, as in raising a heavy body, or it may be the molecular attractions between the particles of a body, as in stamping metal or dividing wood, or it may be friction, as in drawing a heavy body along a rough road.

§ 119. Besides the *effective* resistance, the *effort* is employed in overcoming the *internal* resistances,

chiefly due to friction, which always exist between the different parts of a machine. The effort may be just sufficient to overcome these two kinds of resistance; it may be in excess of what is necessary, or it may be too small. If just sufficient, the machine once in motion will remain uniformly so, or if it be at rest it will be on the point of moving, and the force applied, or *effort*, will be in equilibrium with the effective and internal resistances. If the effort be in excess, the machine will be set in motion and will continue to move with an accelerated motion; if the effort be too small, it will not be able to move the machine; and if the machine be already in motion, it will gradually come to rest. When the effort is just sufficient to overcome the resistance, the ratio of the resistance to the effort is called the *modulus* of the machine. In this case it is evident, from the law of the Conservation of Energy, that the work done by the effort has its equivalent in the work done against the effective resistance to be overcome and against the resistance due to the friction between the parts. This is the general law of machines in uniform motion or in equilibrium, and may be formulated thus: Let P be the force of the effort put forth, p the distance through which it acts in a given time in its own direction, W the external resistance overcome, and w the distance through which it is overcome in the same

time; let S be the resistances due to friction and other causes, s the distance through which these are overcome; then the work done by P equals work done against W and S,

or $$P p = W w + S s.$$

The above equation shows us that whilst P can be made as small as we please by taking p great enough, the mechanical advantage of diminishing P is restricted by the fact that s increases with p; and consequently, with the decrease of P there is a corresponding increase of the work to be done againt friction and the internal resistances. Hence, if friction be neglected, there is no practical limit to the ratio of P to W; but if the friction between the parts be considered—and it is impossible to construct a machine without friction and internal resistance—the advantage of decreasing P has a limit, since if $P p$ remains the same, $W w$ must decrease as $S s$ increases; in other words, the work done against friction increases with the complexity of the machinery.

It will be seen from the above equation that $\frac{W w}{P p} = 1 - \frac{S s}{P p}$, i.e. the ratio of the effective work done, $W w$, to the work applied, $P p$, is a fraction less than unity. This ratio is called the *efficiency* of the machine, and as it approaches unity so the efficiency of the machine increases.

§ 120. In the application of this principle to those simple machines generally known as the *Mechanical Powers*, we shall follow the usual custom of not taking into consideration the resistance due to friction, but shall suppose $S=0$, in which supposed case the *efficiency* of the machine is $\frac{Ww}{Pp}$, and $Pp=Ww$, or $\frac{W}{P}=\frac{p}{w}$. This equation shows that the ratio of the *effective resistance*, which we call simply the *resistance*, to the *effort* is equal to the ratio between the distance through which the effort acts, and the distance through which the resistance is overcome, in the same time; or, in other words, is equal to the ratio of the velocities of the points of application of the effort and resistance. Or, again, if we take equal distances and uniform motion, we see that *what is gained in force is lost in time*. When W is greater than P the machine is said to work at a mechanical advantage, when less at a disadvantage; but in both cases the ratio $\frac{W}{P}$ is called the *mechanical advantage* of the machine. Machines are frequently employed for enabling a large force to overcome a smaller force through a greater distance than the larger force itself acts. With respect to the force applied, such machines work at a mechanical disadvantage; but, with respect to the distance traversed, they work at an ad-

vantage. The advantage or disadvantage of a machine depends, therefore, on the object to be gained. The steam engine and the watch are both machines in which the effort exerted exceeds the resistance to be overcome, but the equation of work holds good.

§ 121. **Virtual Velocities.**—When a frictionless machine is in uniform motion $Pp = Ww$, or $Pp - Ww = 0$; i.e. the algebraic sum of the works done by all the forces is zero. If the machine be in equilibrium the principle of work equally holds good. For, if we suppose the machine to undergo a small displacement, consistent with the connection of its several parts, then, since the forces balance one another, the algebraic sum of the works done by the forces must be zero. In this case, the velocities, being imaginary, are called *virtual*; and the principle is known as the principle of *virtual velocities*, and may be enunciated thus: If any machine is in equilibrium under the action of forces, and we suppose it subjected to any small displacement, consistent with the connection of its parts, the algebraic sum of the works done by all the forces is zero; and, conversely, if the sum be zero, the forces are in equilibrium.

§ 122. **Simple Machines.**—The simple machines, sometimes called the *Mechanical Powers*, may be considered under three heads, according as the

means employed to alter the magnitude and direction of the force applied are—

I. A solid body movable about a fixed point or axis.

II. A flexible string.

III. A hard inclined surface.

Under the first head are comprised (1) the *lever* and (2) the *wheel and axle*; under the second head (3) the various kind of *pulleys*; under the third head (4) *the inclined plane*, (5) the *wedge*, and (6) the *screw*.

§ 123. **The Lever.**—A rigid rod turning on a fixed point or axis is called a lever. The fixed axis is called the *fulcrum*, and those parts of the rod on either side of the fulcrum are called the *arms*.

Let $A\ B$ be a lever, consisting of a uniform rigid rod, supposed to be weightless, turning freely

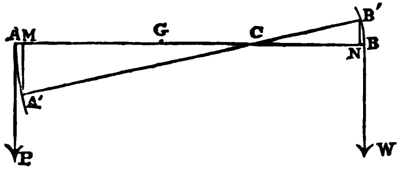

FIG. 30

about C, the fulcrum, and let P be the force applied at A, and W the force exerted, or resistance over-

come, or weight raised at B. Suppose the lever turned through the angle $A\,C\,A'$ into the position $A'\,C\,B'$; then if $A'\,M$, $B'\,N$ are drawn parallel to the directions of P and W, the distance through which the point A has moved in the direction of P is $A'\,M$, and the corresponding distance through which the point B has moved in the direction opposite to W is $B'\,N$. We have, therefore, by the equation of work,

$$P \times A'\,M = W \times B'\,N;$$

and since, by similarity of triangles,

$$A'\,M : B'\,N :: A'\,C : B'\,C,$$

and $$A'\,C = A\,C, \text{ and } B'\,C = B\,C,$$

we have $$P \times A\,C = W \times B\,C,$$

or $$P \times \text{its arm} = W \times \text{its arm}.$$

This is the general principle of the lever, when the bar is a straight rod and the forces are perpendicular to it.

§ 124. **Three Kinds of Levers.**—Levers are said to be of three kinds, according to the position of the fulcrum with respect to the *effort*, or force *applied*, and the *resistance*, or force *overcome*.

Where the *fulcrum* is in the middle the lever is of the first kind; where the point of application of the *resistance* is in the middle, the lever is of the second kind; where the point of application

of the *effort* is in the middle, the lever is of the third kind.

In each of these three cases

$$\frac{W}{P} = \frac{AC}{BC}.$$

In case I. there is a mechanical advantage or disadvantage, as AC is greater or less than BC.

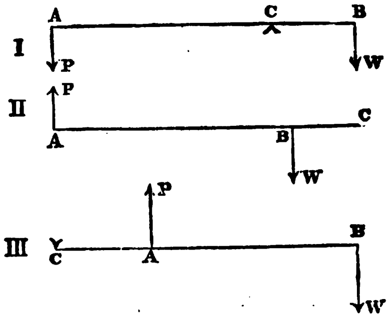

FIG. 31

In case II. AC is necessarily greater than BC, and there is always a mechanical advantage.

In case III. AC is necessarily less than BC, and there is always a mechanical disadvantage.

Besides the forces P and W, there is the reaction of the fixed axis or fulcrum to be considered, which is equal to the total force with which the bar presses on the fulcrum. Neglecting the weight of the bar, it is evident that

in case I. the reaction is equal to $P + W$,
in case II. „ „ $W - P$,
in case III. „ „ $P - W$.

By means of a lever two forces, however unequal, may be made to balance each other, by so adjusting the position of the fulcrum that its perpendicular distances from the directions of the forces shall have the inverse ratio of the forces.

§ 125. **Examples of Levers.**—The crowbar is a good example of a lever of the first kind. The wheelbarrow is a lever of the second kind in which the fulcrum is at the axle of the wheel, and the effort is applied at the handle. A pair of nut-crackers is a double lever of the second kind. The oar of a rowing-boat may be regarded as a lever of the first kind, in which the fulcrum is at the rowlocks, if we consider the resistance of the water as the force to be overcome. The treadle of a sewing-machine is an example of a lever of the third kind. The best examples of this order of levers are seen in the animal frame, in which readiness of action is obtained at a loss of power.

§ 126. **Weight of the Lever.**—If the weight of the lever be taken into account, we have an additional force to consider, which may assist the action of the force applied or give the force more work to do, according as it acts on the same side

of the fulcrum as this force or not. If G (fig. 30) be the position of the point in the lever, in which the single force equal to the weight may be supposed to act (and it will be shown in a subsequent chapter that such a point exists in, or in respect to, every heavy body), and if w be the weight of the lever acting at this point, then the equation of work becomes

$$P \times AC + w \times GC = W \times BC,$$

or $P \times AC = W \times BC + w \times GC$, according to the position of G with respect to C.

§ 127. **Examples.**—(1) A lever is 3 feet long, and its weight is 1 lb., and acts at the middle point. The resistance to be overcome at one end is 20 lbs., and the force applied at the other end is 3 lbs.; find the position of the fulcrum.

Let x be the distance of the fulcrum from A where P acts (fig. 30); then $3 - x$ is the distance of the fulcrum from B where W acts, and $x - \frac{3}{2}$ is the distance of the fulcrum from G where w acts;

$$\therefore\ P \times x + w \times \left(x - \frac{3}{2}\right) = W \times (3 - x),$$

$$\therefore\ 3x + x - \frac{3}{2} = 20\,(3 - x) = 60 - 20x,$$

$$\therefore\ x = 2\tfrac{9}{16} \text{ feet} = \text{the distance of fulcrum from } A.$$

(2) What force must be applied at the end of a lever 20 inches long to lift a body weighing 30 lbs. slung at a point 5 inches from the other end, if the weight of the lever is 4 lbs. and acts at its middle point?

Here $P \times AC = 4 \times CG + 30 \times CB,$

or $20P = 4 \times 10 + 30 \times 5;$

$$\therefore P = 9\tfrac{1}{2} \text{ lbs.}$$

The various kinds of balances are examples of levers; but these will be treated later on, under the head of *Moments*, when the principle of levers will be further considered.

EXERCISES XIII[1]

1. A lever is 18 inches long; where must the fulcrum be placed in order that a weight of 6 lbs. at one end may balance double its weight at the other end?
2. What force must be applied at one end of a lever 12 inches long to raise a mass weighing 30 lbs. hanging 4 inches from the fulcrum which is at the other end, and what is the force pressing on the fulcrum?
3. Two bodies weighing 6 lbs. and 8 lbs. respectively are hung from the ends of a lever 7 feet long; where must the fulcrum be placed so that they may balance?
4. A mass weighing 10 oz. at the end of a lever is raised by a force which is just greater than the weight of 36 oz., and which acts 6 inches from the fulcrum which is at the other end; what is the length of the lever and the force pressing on the fulcrum?
5. A lever weighs 3 oz., and its weight acts at its middle point; the ratio of its arms is 1 : 3. If a body weighing 48 oz. be hung from the end of the shorter arm, what is the weight of the mass that must be suspended from the other end to prevent motion?
6. A lever 10 inches long, the weight of which is 4 oz. and acts at its middle point, balances about a cer-

[1] In these and the following exercises the forces are given in gravitation units unless otherwise stated.

tain point when a body weighing 6 oz. is hung from one end; find the point.

7. A heavy lever weighing 8 oz. balances at a point 3 inches from one end and 9 inches from the other. Will it continue to balance about that point if masses of equal weight be suspended from the extremities?
8. A beam the length of which is 12 feet balances at a point 2 feet from one end; but if a body weighing 100 lbs. be hung from the other end, it balances at a point 2 feet from that end; find the weight of the beam.
9. A beam the length of which is 8 feet balances at a point 2 feet from one end. If to this end a body weighing 40 lbs. be hung, find the least force applied at the other end that will support this body.
10. A heavy beam 16 feet long, and weighing 4 lbs., balances by itself about a point 4 feet from one end. If a body weighing 10 lbs. be hung 2 feet from this end, find the weight of the body that must be hung from the other extremity that the beam may balance about its middle point.

XVII. *The Wheel and Axle and Toothed Wheels*

§ 128. **Wheel and Axle.**—The wheel and axle is a modification of the lever. It consists of two cylinders having a common axis, the larger of which is called the wheel and the smaller the axle. The axis common to both is horizontal. A mass of weight W (fig. 32) hangs at the end of a string fastened to the axle, and the string is coiled round the axle by the revolution of the wheel. The wheel may

be moved by the hand or by a string with a weight applied to it.

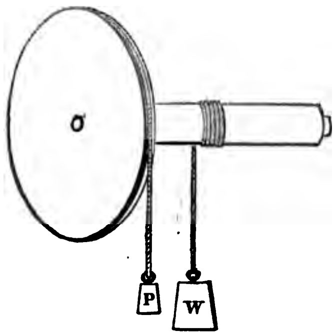

FIG. 32

Viewed in section we have the force P (fig. 33) acting at the end of the radius of the large wheel at A, and the force W at the end B of the radius of the

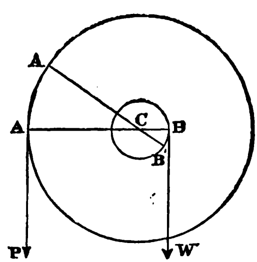

FIG. 33

small wheel or axle. If the wheel move through an arc $A\ A'$, the point of application of P moves from A' to A, and that of W moves in the same

time from B' to B. Then the work done by P is $P \times AA'$, and the work done by W is $W \times BB'$,

$$\therefore P \times AA' = W \times BB',$$

or
$$\frac{W}{P} = \frac{\text{arc } AA'}{\text{arc } BB'} = \frac{AC}{BC'}$$

i.e. $\frac{W}{P} = \frac{\text{radius of wheel}}{\text{radius of axle}} = \frac{\text{circumference of wheel}}{\text{circumference of axle}}$.

§ 129. Examples of the wheel and axle are very numerous. One of the commonest is the

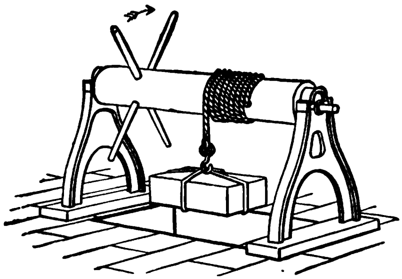

FIG. 34

FIG. 35

windlass (fig. 34), in which the force is applied at the and of a handle. In the capstan (fig. 35) the axle is

vertical, and the mechanical advantage is increased by the length of the pole inserted into the circumference. The various kinds of mills are examples of this machine, in which the resistance to be overcome is mainly due to friction.

§ 130. **Toothed Wheels.**—Wheels having teeth that fit into one another do the work of levers acting continuously. The mechanical principle of these wheels is the same as that of the lever or wheel and axle.

Suppose we have two wheels, as in fig. 36, the teeth of which exactly fit into one another, and that the wheel M is worked by a force P, acting at right angles to a lever at its end B, whilst the axis of the larger wheel H supports a mass of weight W. Then the ratio of W to P can be easily found by the application of the principle of work. Since the teeth of both wheels are of the same size, it follows that for every complete revolution of the wheel H the wneel M will make as many revolutions as the number of times the circumference of H is greater than the circumference of M. If $C =$ the circumference of H, or the number of teeth in H, and $c =$ the circumference of M, or number of teeth in M, then for every revolution of H, M will make $\frac{C}{c}$ revolutions. But for every revolution of H, the point at which W

acts is moved through a distance equal to the circumference of the axis, and for every revolution of M the point B, where P is applied, moves through the circumference of the circle described by B. Hence the work done by W in one revolu-

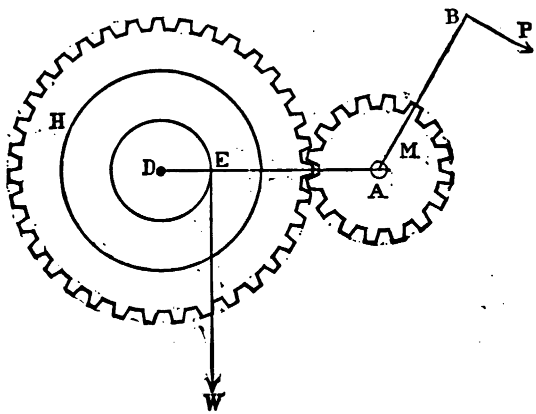

FIG. 36

tion of $H = W \times 2\pi . DE$, and the work done by P in the same time is $P \times 2\pi AB \times \frac{C}{c}$;

$$\therefore\ W \times 2\pi . DE = P \times 2\pi AB \times \frac{C}{c},$$

$$\therefore\ W \times DE = P \times AB \times \frac{C}{c},$$

or $$\frac{W}{P} = \frac{AB}{DE} \times \frac{\text{No. of teeth in } H}{\text{No. of teeth in } M}.$$

Another simple application of the toothed wheel is known as the rack and pinion. It consists of a small wheel, with cogs as teeth, that is made to work into a vertical bar likewise filled

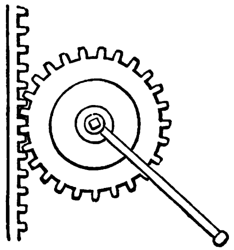

FIG. 37

with teeth. In this way motion round an axis is converted into motion in a straight line. The piston of a double-barrelled air-pump is worked by such a contrivance.

EXERCISES XIV

1. What is the diameter of a wheel if a force equal to a weight of 3 oz. is just able to move a mass of 12 oz. that hangs from the axle, the radius of the axle being 2 inches?
2. What is the mechanical advantage of a wheel and axle in which the diameter of the axle is 3 inches, and the radius of the wheel 12 inches?
3. Is the mechanical advantage of the wheel and axle increased or diminished by lessening the radii of wheel and axle by the same amount?
4. If a mass of 20 oz. be supported on a wheel and axle by a force equal to the weight of 4 oz., and the radius of the axle is $\frac{3}{8}$ inch, find the radius of the wheel.

5. A capstan is worked by a man pushing at the end of a pole. He exerts a force of 50 lbs., and walks 10 feet round for every 2 feet of rope pulled in. What is the resistance overcome?
6. A man, whose weight is 140 lbs., is just able to support a heavy body that hangs over an axle of 6 inches radius by hanging to the rope that passes over the corresponding wheel, the diameter of which is 4 feet; find the weight of the body supported.
7. If the radii of wheel and axle be as 10 is to 4, and masses of 3 oz. and 8 oz. hang from them, which will descend?
8. If the difference between the diameter of a wheel and the diameter of the axle be six times the radius of the axle, find the greatest weight that can be sustained by a force of 60 lbs.
9. If the radius of the wheel be n times as great as that of the axle, and t be the maximum pull of the string on the wheel, find the greatest weight that can be supported.
10. If the radius of the wheel is three times that of the axle, and the cord round the wheel can support a weight not exceeding 40 lbs., find the greatest mass that can be lifted.
11. In the wheel and axle is there any advantage in having the rope that passes round the wheel thicker than that which passes round the axle?
12. The radius of the wheel being three times that of the axle, and the cord on the wheel being only strong enough to support a pull equivalent to 30 lbs. weight, find the greatest mass which can be lifted.

XVIII. *The Pulley*

§ 131. The pulley consists of a circular plate or disc, the circumference of which is generally

grooved to receive a cord which passes over it. The pulley is made to revolve freely about an axis, working in bearings, fixed into a framework called the block. When the position of the axis

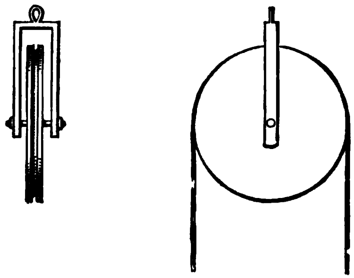

FIG. 38

of the pulley is fixed, the pulley is called a *fixed* pulley; but where the block, and with it the axis, can ascend or descend, it is called a *movable* pulley.

§ 132. **Fixed Pulley.**—The fixed pulley can only change the direction of a force. Wherever it is required to change a pushing into a pulling force, the fixed pulley can be advantageously employed. It works, however, without any mechanical advantage in the technical sense in which the term is employed. The assumption involved in all elementary calculations with respect to the pulley is the constancy of the stress in all parts of the same string when transmitting a force

(§ 85). This is equivalent to the neglect of all friction between the pulley and the string, as well as the internal friction of the string when it is being bent to pass round the pulley. It should be observed that in the pulley, as in other mechanical appliances, advantage is taken of the internal molecular stresses that are called into

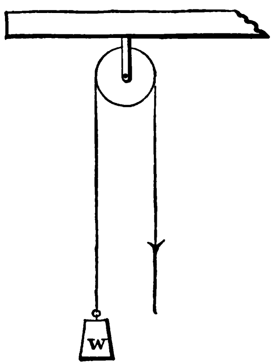

FIG. 39

play whenever cords, beams, &c., are subjected to strain by external forces. Thus, the pulley requires to be attached to a beam, and this beam must be strong enough to support the force exerted by the weight and the force employed to raise it. In the problems we shall consider respecting the pulley the string is supposed to be perfecly flexible and inextensible, and the friction is neglected.

§ 133. **The Single Movable Pulley.**—A cord is

fixed to a beam at A, and passes under the movable pulley $B\ D$, to which a body weighing W is attached. The other end of the cord passes over a fixed pulley C, and supports the weight P. Now, it is clear that for every inch $B\ D$ rises, the point at which P acts will descend two inches, and consequently the equation of work gives us $W = 2\,P$ or $P = \frac{W}{2}$.

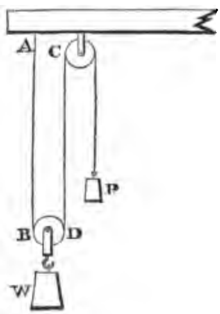

FIG. 40

The same result may be arrived at by supposing the machine at rest and P supporting W. In this case, since we neglect friction, the stretching force of the cord is the same throughout, and the weight W is supported by the two strings $A\ B$, $C\ D$, in each of which the pull is P, we have $W = 2\,P$ or $\frac{W}{P} = 2$.

The mechanical advantage with a single movable pulley, therefore, equals 2, if the cords be parallel.

The single movable pulley may be used with other combinations. Thus, if A and B (fig. 41) be two fixed pulleys separated by an interval equal to the diameter of either, and if C be a pulley of twice the diameter of A or B, and if one end of the cord be fixed to C, whilst the other, after passing over the three pulleys, supports a weight P, then, whether the machine be in uniform motion or at rest, $W = 3P$.

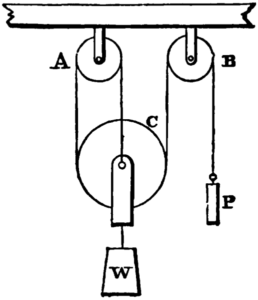

FIG. 41

If a man be suspended from a movable pulley, and support himself by holding on to the other end of the cord, the downward force on the movable pulley is *less* than his real weight by the force with which he pulls, so that if W be his actual weight, and P the force with which he pulls on the loose end of the cord, the downward force on the pulley to which he is attached is $W - P$, and since this force is supported by P in each of the

two cords holding up the movable pulley, $W - P = 2P$ or $W = 3P$. That is, he pulls with a force equal to one-third of his weight.

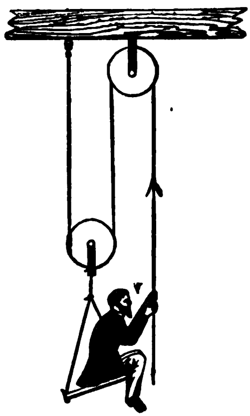

FIG. 42

§ 134. As regards the force acting on the beam, we have, in the case represented in fig. 40, a force of P at A, and a force of $2P$ at C, or the total force $= 3P = W + P$. In fig. 41 we have $2P$ at A, and $2P$ at B, or total force $= 4P$; and in fig. 42 we have the total force $= 3P = W$.

§ 135. **Combinations. First system, in which each Pulley hangs by a separate cord.**—Each cord has one end attached to a fixed point in the beam, and all, except the first, have the other end

attached to a movable pulley. The diagram explains the arrangement.

It is clear that if the lowest pulley on which W acts ascend 1 inch, the next pulley rises 2 inches, the next 4 inches, and the mass weighing P descends

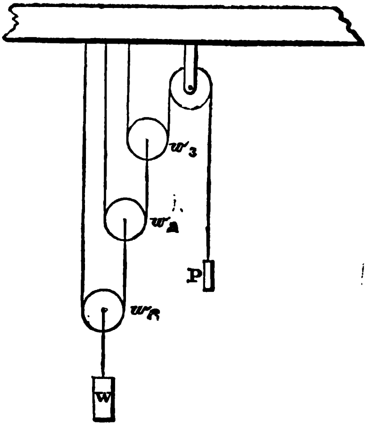

FIG. 43

8 inches. Thus, with three movable pulleys, the equation of work gives us $W = 8\ P = 2^3\ P$. If there be n *movable* pulleys $W = 2^n\ P$ or $\frac{W}{P} = 2^n$.

Applying the principle of the constancy of the pulling force in each cord, we see that the pull in the cord on which P acts is P throughout; in the cord immediately beneath this it is $2\ P$ throughout, in the next $4\ P$, and the

double pull of $4P$ supports W. In other words, $W = 8P$.

The force acting on the beam in this case is $W + P = 9P$, and if the cord when it leaves the highest movable pulley be at once supported by a force P without passing over a fixed pulley, the force acting on the beam is $7P$, and the weight W is jointly supported by this force and the force P.

§ 136. We shall now consider what alterations must be made if the weights of the movable pulleys be taken into account.

Let w_1 be the weight of the lowest pulley.

Let w_2 " " next " and so on.

Then the distances the pulleys are moved through being those given above, the equation of work is

$$8P = W + w_1 + 2w_2 + 4w_3;$$

or $$P = \frac{W}{2^3} + \frac{w_1}{2^3} + \frac{w_2}{2^2} + \frac{w_3}{2}.$$

§ 137. **Example.**—With three movable pulleys, arranged as above, each of which weighs 8 oz., what weight can be supported by a mass weighing 1 lb.?

Here $$P = \frac{W}{8} + \frac{w}{8} + \frac{w}{4} + \frac{w}{2} = \frac{W}{8} + 7,$$

$$\therefore 16 \text{ oz.} = \left(\frac{W}{8} + 7\right)\text{oz.},$$

$$\therefore W = 72 \text{ oz.}$$

§ 138. **Second System, in which the same cord passes round all the Pulleys**.—Figs. 44 and 45 show two methods of arranging this system. Suppose there are three pulleys in each block,

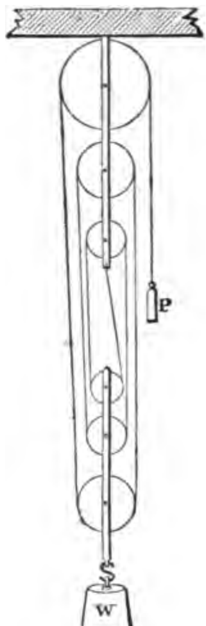

FIG. 44

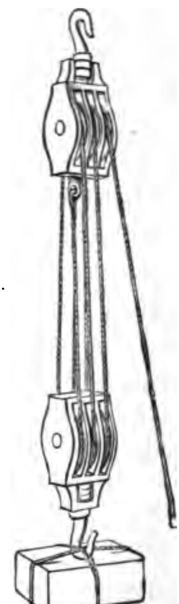

FIG. 45

then it is clear that if the body of weight W ascend 1 inch, each of the cords between the blocks must be shortened 1 inch, and therefore the point where P acts descends 6 inches, or $W = 6\ P$. Again,

since the pull of the cord is the same in every part of the cord, and is everywhere equal to P, and since we have six cords supporting W, it is clear, supposing all the cords to be parallel, that $W = 6P$.

If we have n pulleys in the two blocks, and w is the weight of the lower block, $W + w = nP$. The force pulling on the beam is $W + P +$ weights of both blocks. This system is most commonly employed on account of its superior portability. The several pulleys are generally mounted on a common axis, and enclosed in a single block, as shown in fig. 45.

§ 139. **Third System, in which each cord is attached to the block supporting the weight.**— In this system one end of each cord is attached to the bar on which the weight acts, and the other supports a pulley. The block of the highest pulley is attached to the beam. The force P acts at the unattached end of the cord DA.

To apply the principle of work, we see that when the bar is raised *one* inch the cord CF is shortened *one* inch, in consequence of which the pulley B moves down one inch, and the cord BE is shortened *two* inches. The pulley A, therefore, moves down *three* inches, two inches through the fall of B, and one inch through the rise of the bar. For a similar reason the point where P acts moves

down *seven* inches, six inches in consequence of the fall of A through three inches, and one inch through the rise of the bar supporting W;

$$\therefore W = 7P.$$

The same result may be more easily obtained

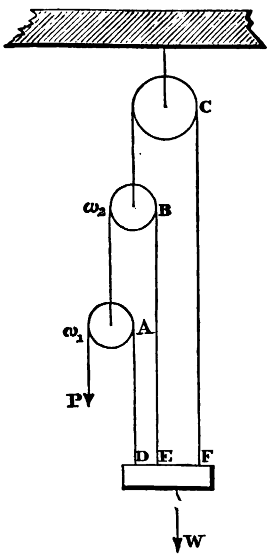

FIG. 46

by considering the pull in the cords. The weight W is supported by the pulling forces in the three cords DA, EB, FC.

These forces are respectively equal to P, $2P$,

and $4P$, $\therefore W = P + 2P + 4P = 7P$. If there be n pulleys,

$$W = P + 2P + 2^2 P + 2^3 P + \dots + 2^{n-1} P$$
$$= (2^n - 1) P, \text{ as shown in books on algebra;}$$

$$\therefore \frac{W}{P} = 2^n - 1.$$

If the weights of the pulleys be considered we have :—

$$\begin{aligned} &\text{Pull in the cord } AD = P, \\ &\quad ,, \qquad ,, \quad EB = 2P + w_1, \\ &\quad ,, \qquad ,, \quad CF = 4P + 2w_1 + w_2; \end{aligned}$$

and the sum of all these pulling forces equals W,

$$\therefore W = 7P + 3w_1 + w_2.$$

In this system the weights of the *movable pulleys assist* P; in the two former systems they act against it.

EXERCISES XV

1. A man weighing 140 lbs. pulls up a mass weighing 80 lbs. by means of a fixed pulley under which he stands; find the force, in gravitation units, with which he presses on the floor.
2. Find the force which will support a weight of 600 lbs., with three movable pulleys arranged as in the first system.
3. What force is necessary to raise a body weighing 120 lbs. by an arrangement of six pulleys in which the same string passes round each pulley?

4. If, in the preceding problem, the weight of the block be 6 lbs., what additional force will be required?
5. Find the 'mechanical advantage' of a system of three movable pulleys arranged as in the annexed diagram.

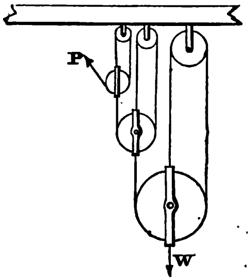

FIG. 47

6. If three movable pulleys, the weights of which are 2 oz., 4 oz., and 8 oz., be arranged as in the first system, what is the least force that will raise a weight of 104 lbs.?
7. If there be equilibirum between P and W with three pulleys in that system in which each string is attached to the load, what additional weight can be supported if 2 lbs. weight be added to P?
8. A man weighing 150 lbs. raises a mass of 4 cwt. by a system of four movable pulleys arranged according to the first system; what is his pressing force on the ground?
9. What would be the difference in the force if each pulley weighed 4 oz.?
10. Find the force necessary to sustain a weight of 100 lbs. with three movable pulleys arranged according to the third system, the weights of the pulleys being 8 oz., 6 oz., and 4 oz. respectively.
11. Find the relation between P and W in a system of

pulleys arranged as in the annexed diagram, supposing the weight of each of the movable pulleys to be the same.

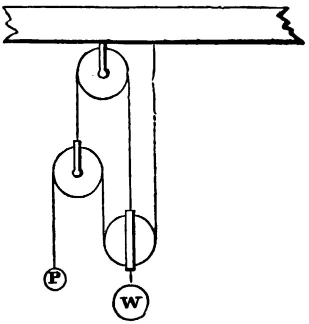

FIG. 48

12. Neglecting the weights of the pulleys, what force would a man, whose weight is 160 lbs., exert on the ground in raising a mass of 500 lbs. by means of the above combination?

13. In a system of one fixed and four movable pulleys, in which one end of each string is fixed to a beam, find the relation between the forces at either end (neglecting the weights of the pulleys), when one of the strings is nailed to the pulley round which it passes.

    What is the force exerted on the beam to which the strings are attached?

14. A man sitting upon a board suspended from a single movable pulley pulls downwards at one end of a rope which passes under the movable pulley and over a pulley fixed to a beam overhead, the other end of the rope being fixed to the same beam. What is the smallest proportion of his whole weight with which the man must pull in order to raise himself?

15. With what force would he require to pull upwards, if the rope, before coming to his hand, passed under a pulley fixed to the ground, as well as round the other two pulleys?

16. In the third system of pulleys, in which each string is attached to the load, each pulley weighs $3\frac{1}{2}$ oz.; find the weight which will be supported by the pulleys alone, when there are five movable pulleys.

17. Suppose that we have four weightless pulleys, three movable and one fixed, forming an example of the first system, and that the load is a man weighing 160 lbs., find what pull the man must exert on the free end of the rope in order to raise himself thereby.

XIX. *The Inclined Plane—Wedge—Screw*

§ 140. By means of an inclined plane, a heavy body can be raised to a certain height by the application of a force less than the weight of the

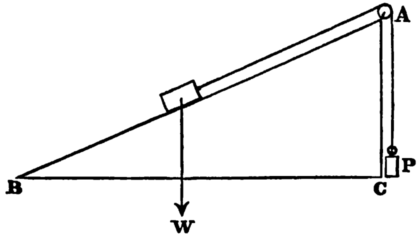

FIG. 49

body. All roads that are not level may be regarded as inclined planes.

If *A B C* (fig. 49) be an inclined plane, then

$A\ B$ is called the *length* of the plane, $B\ C$ the *base* of the plane, and $A\ C$ the *height* of the plane. We shall denote these by l, b, h respectively.

§ 141. **Uniform motion on an Inclined Plane by application of a force parallel to the Plane.**—This case has already been partly considered. If a body the weight of which is W move from B to A by the continued action of a force P, then the work done by P is $P \times A\ B$, and the work done by W against gravity, i.e. in the direction $C\ A$, is $W \times C\ A$, since W has been raised through a vertical distance equal to $C\ A$;

$$\therefore P \times A\,B = W \times C\,A, \text{ or } \frac{W}{P} = \frac{l}{h}.$$

§ 142. **Uniform motion on an Inclined Plane by application of a horizontal or pushing force.**—

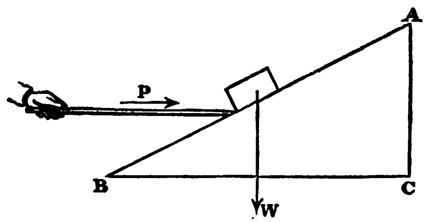

FIG. 50

If the force P act in the direction $B\ C$, the work done by P is $P \times B\ C$,

$$\therefore P \times B\,C = W \times C\,A, \text{ or } \frac{W}{P} = \frac{B\,C}{C\,A} = \frac{b}{h}.$$

§ 143. **Examples.**—(1) Find the weight W which

can be supported by two forces P and Q, one acting parallel to the length and the other parallel to the base of the plane.

Since P alone can support a weight equal to $\frac{l}{h}P$, and Q alone a weight equal to $\frac{b}{h}Q$, if W be the weight supported by both acting together,

$$W = \frac{l}{h}P + \frac{b}{h}Q,$$

or
$$Wh = Pl + Qb.$$

The same result may be directly deduced from the principle of work.

(2) Two inclined planes of different lengths, but having a common height, are placed back to back, and two bodies, weighing P and Q, connected by a cord that passes over a pulley at their summit, are moving *uniformly*, or are at rest, on them; if l and l' be the lengths of the planes, find the ratio of P to Q.

If the two masses are moving uniformly, or are at rest, it is clear that the pull in the cord must be equal to the force just sufficient to support either. Call this pull T, then $T = \frac{h}{l}P = \frac{h}{l'}Q$,

$$\therefore \frac{P}{l} = \frac{Q}{l'},$$

or
$$P : Q :: l : l'.$$

The Wedge.

§ 144. **The Wedge** is a double inclined plane, movable instead of fixed, as in the cases considered, and used for separating bodies. The force is applied

in a direction perpendicular to the height of the plane, i.e. parallel to the base, and the resistance to be overcome consists of the molecular attractions of the particles of the body which are being separated. If the friction between the surfaces of the body and the wedge be entirely neglected, this resistance may be assumed to act in a direction at right angles to the inclined surface of the wedge, or length of the plane. This assumption, it must be understood, is, however, very far removed from

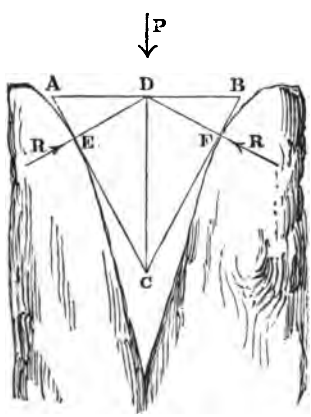

Fig. 51

what actually takes place. The line $A\ B$ is called the back of the wedge.

Suppose the wedge has been driven into the material a distance equal to $D\ C$ by a force P acting

in the direction $D\,C$, then it is clear that the work done by P is $P \times D\,C$. Draw $D\,E$, $D\,F$ perpendicular to $A\,C$, $B\,C$. Then, since the points E and F were originally together, the work done against the resistance R is $R \times D\,E + R \times D\,F = 2\,R \times D\,E$. Hence the equation of work gives

$$P \times D\,C = R \times 2\,D\,E \therefore \frac{R}{P} = \frac{D\,C}{2\,D\,E}.$$

But $\dfrac{D\,C}{D\,E} = \dfrac{A\,C}{A\,D}$ by similarity of triangles,

$$\therefore \frac{R}{P} = \frac{A\,C}{2A\,D} = \frac{A\,C}{A\,B} = \frac{\text{length of one of the equal sides}}{\text{back of the wedge}}.$$

This shows that as the size of the back of the wedge is lessened the mechanical advantage of the wedge is increased. Knives, choppers, chisels, and many other cutting implements are examples of the wedge.

§ 145. In the action of the wedge a great part of the energy of the force applied is employed in cleaving the material into which it is driven. The force required to effect this is so great that instead of applying a continuous pushing force perpendicular to the back, a series of blows is generally given to it. In this way a large amount of energy is directed for an indefinitely short period of time against the molecular attractions of the body. When the resistance to be overcome is very great a series of *impulses* or *blows* is more

effectual than a continuous moving force. A certain amount of energy is stored up in a descending hammer, which is at once applied when the blow is given.

The Screw.

§ 146. **The Screw** is a machine which is sometimes employed to overcome resistance and sometimes to multiply pressure. It consists of a cylinder

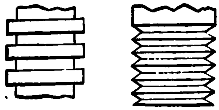

FIG. 52

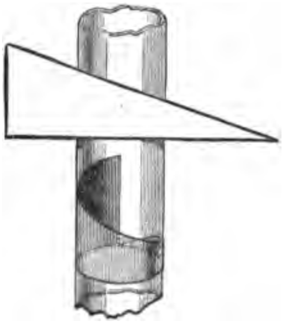

FIG. 53

with a uniform projecting thread traced round its surface and inclined at a constant angle to lines parallel to the axis of the cylinder. The thread

of the screw may be formed by wrapping an inclined plane round the cylinder, as shown in fig. 53, the base of the plane corresponding with the circumference of the cylinder, and the height of the plane with the pitch, or the distance between the threads. The threads are of different shapes: they may be square or V-shaped, as shown roughly in fig. 52. The force is usually transmitted by connecting the screw with a concave cylinder, called the nut, having a spiral cavity on its inner surface, corresponding to the spiral projections.

The force or *effort* is almost always applied as in the screw press, fig. 54, at the end of a lever fixed to the centre of the cylinder. It is evident that a screw never requires any force in the direction of its axis. The cylinder must be made to revolve only; and this can be effected by a force acting at right angles to the extremities of its diameter, or of its diameter produced.

If, as in the screw press, a force P acts at A horizontally, and at right angles to $A\,B$, and the cylinder of the screw $C\,D$, which works in the nut E, make one revolution, the point of application of P will move through a distance equal to the circumference of the circle, of which $A\,B$ is the diameter; and the work done by P will be $P \times$ circumference of circle described by $A\,B$. During

the same time the screw will move in the direction of its axis through a distance equal to the pitch of the screw, and this is the direction in which

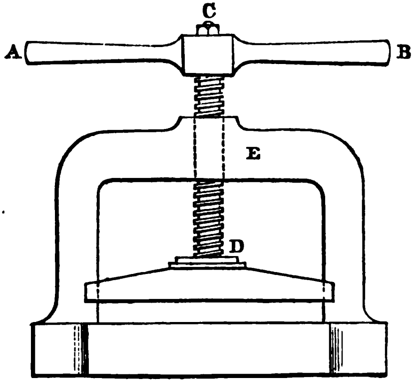

FIG. 54

the resistance is encountered. Hence the work done against the resistance W is $W \times$ distance between the threads.

If C be the circumference described by the extremity of P's arm and d the pitch or distance between two adjoining threads, and if P and W be respectively the force applied and the pressure produced,

$$P \times C = W \times d, \text{ or } \frac{W}{P} = \frac{C}{d}.$$

Usually the force P is applied both at A and B, in which case $\frac{W}{P} = \frac{2\,C}{d}$.

EXERCISES XVI

1. What weight can be supported on a plane by a horizontal force of 10 poundals, if the ratio of the height to the base is $\frac{3}{4}$? ($g=32$.)
2. With a force of 3 oz. weight acting parallel to the plane, what weight can be supported on a plane that rises 2 in 7?
3. A plane rises 3 in 8; what force parallel to it is required just to move a mass of 10 oz.?
   A plane rises 2 in 7; what force parallel to it is required just to move a mass of 12 oz.?
   A plane rises 5 in 9; what force parallel to it is required just to move a mass of 16 oz.?
   A plane rises $1\frac{1}{4}$ in $12\frac{1}{2}$; what force parallel to it is required just to move a mass of 20 oz.?
4. Find the inclination of the plane if a horizontal force of 5 kilograms weight can just move a mass of 12 kilograms.
5. The inclination of a plane is 30°, and at the summit is a smooth wheel over which passes a fine thread. At one end of the thread is a mass of 10 oz. resting on the plane, and from the other hangs a heavy body supporting it. Find the pull in the thread.
6. Find the horizontal force necessary to support a body weighing 1 lb. on a plane that rises 3 in 5.
7. The angle of a plane is 45°; what weight can be supported by a horizontal force of 3 lbs. and a force of 4 lbs. parallel to the plane, both acting together?
8. Two planes, having the same height, are placed back to back, and two masses of 7 lbs. and 10 lbs. connected by a string passing over the summit move uniformly upon them; find the ratio of the lengths of the planes.
9. A body weighing 7 lbs. is supported on a plane that rises 1 in 7 by a force that acts parallel to the plane;

if the mass of the body be increased by 3 lbs., what force will be required to support it?

10. A heavy body is just able to be pulled up a plane by a force parallel to it and equal to $\frac{1}{5}$ of the weight of the body; find the ratio of the height to the base of the plane.

11. Two masses hang over a pulley fixed to the summit of a smooth inclined plane, on which one mass is supported, and for every 3 ins. that the one is made to descend the other rises 2 ins.; find the ratio of the masses and the length of the plane, the height being 18 ins.

12. If a weight W be supported on an inclined plane by a force $\frac{W}{2}$ parallel to the plane, what is the inclination of the plane?

13. The ratio of the height of a plane to its length is 2 : 15; what horizontal force is necessary to support a weight of 10 lbs.?

14. The inclination of a plane is 30°; a mass of 80 lbs. being placed upon it, a force of 63 lbs. is required to pull the mass up the plane; what is the coefficient of friction?

15. Two inclined planes, of the same height, one of which is 8 ft. long, and the other 5 ft., are placed so as to slope in opposite directions and so that their summits coincide. A mass of 20 oz. rests on the shorter plane, and is connected by a string passing over a pulley at the common summit of the two planes with a mass resting upon the longer plane; how great must be the weight of this mass to prevent motion?

16. In a screw which has seven threads to the inch, find the resistance that can be overcome by a force of 60 poundals applied at the circumference, the radius of the cylinder being 1 in.

17. If the circumference of a screw be 10 ins., what force must be applied to overcome a resistance of 30 lbs.

weight, the distance between the threads being $\frac{1}{4}$ in.?

18. How many turns must be given to a screw formed upon a cylinder whose length is 10 ins., and circumference 5 ins., that a force of 2 units may overcome a resistance of 100 units?
19. A screw is made to revolve by a force of 2 units applied at the end of a lever 3·5 ft. long; if the distance between the threads be $\frac{1}{2}$ in., what reaction can be overcome?
20. The circumference described by the point to which the force is applied is 4 ft., and the distance between the threads is $\frac{1}{4}$ in.; what force is required to overcome a resistance of 1 ton?

EXAMINATION QUESTIONS VI

1. A wheel and axle is used to raise a bucket from a well. The radius of the wheel is 15 ins., and while it makes seven revolutions the bucket, which weighs 30 lbs., rises $5\frac{1}{2}$ ft. Show what is the smallest force that can be employed to turn the wheel. Upon what general principle is your answer founded?—*Univ. of Lond. Matric.*, Jan. 1872.
2. Ten weights, each of 20 lbs., are to be lifted to a height of 8 ft. from the ground. Show how a system of pulleys might be arranged so that, disregarding friction and the weight of the pulleys, all the weights could be lifted together by exerting a force equal to one of them. Show that the distance through which this force would have to act would be the same as when the weights were raised one by one by the same power.—*Ib.* Jan. 1873.
3. Make careful sketches of (α) a system of (weightless) pulleys in which 1 lb. balances 32 lbs., and (β) a system of (weightless) pulleys in which 1 lb. balances 15 lbs., taking care in each case that the

number of pulleys is the least possible.—*Ib.* July 1879.

4. A screw whose pitch is $\frac{1}{4}$ in. is turned by means of a lever 4 feet long; find the force which will raise 15 cwt.—*Ib.* Jan. 1882.

5. Find the relation between the forces P and W in a system of five movable pulleys in which each pulley hangs by a separate string and the weight of each pulley is equal to P.—*Ib.* June 1884.

6 Describe and sketch a system of pulleys on which (neglecting the weight of the pulleys) a 'power' (force) of $10\frac{1}{2}$ lbs. would balance a 'weight' of 84 lbs., and show how far the 'power' must move in order to raise the weight 3 ft.—*Ib.* June 1885.

7. In the system of pulleys in which each pulley hangs by a separate string, how would you find experimentally the relation between the 'tension' of any string and that of the string next above it? If there be four pulleys in the system, and each weigh 2 lbs., what weight can be raised by a 'power' (force) equal to the weight of 20 lbs.?—*Ib.* Jan. 1888.

8. Friction is neglected, and it is found that a force acting horizontally will move 10 lbs. up 5 feet of an incline rising 1 in 4. Find the work done and also the force parallel to the plane which will just support it.—*S. & A. Dept.*

9. A rod 7·5 feet long can move freely in a plane round one end; it is acted on at right angles to its length by a force of 35 lbs.; find the number of foot-pounds of work done by the force in one turn. How many turns must it take a minute if the force works with one horse-power? ($\pi = 3\frac{1}{7}$.)—*Ib.* 1888.

10. A man exerts a pressure of 30 lbs. on the arm of a capstan at a distance of 10·5 feet from the centre of rotation; he works at the rate of 198,000 foot-pounds an hour. How many times does the capstan turn in an hour?—*Ib.* 1890.

11. In the first system of pulleys find the force necessary

to support a weight of 4,000 lbs. when there are four movable pulleys. Find also the pull of each rope on the beam, and if the sum is not equal to the weight explain the difference (friction and weights of ropes and pulleys are to be neglected).—*Ib.* 1890.

12. Let $A\,B$ be a horizontal line 10 feet long, and F a point in it 6 inches from A; suppose that $A\,B$ is a lever that turns on a fulcrum under F, and carries a weight of 50 lbs. at B; if it is kept horizontal by a fixed point above the rod 5 inches from F and 1 inch from A, find the pressure on the fulcrum and on the fixed point.—*Ib.* 1891.

STATICS—REST

CHAPTER VII

THEORY OF EQUILIBRIUM

XX. *Introduction—Forces in the same Straight Line*

§ 147. **Problem of Statics.**—The problem of statics is to determine the conditions under which several forces acting on a body produce equilibrium. To solve this problem it is often convenient to find the single force, if one exists, that will produce the same effect as the other forces taken together. This single force, which can replace several other forces, is called the *resultant*, and the forces themselves are called *components*. The resultant of the acting forces being determined, the problem of statics is solved when the force is found which will keep this resultant in equilibrium. We shall be occupied, therefore, for some time in finding the resultant of forces acting in various directions.

§ 148. **Method of estimating and representing Statical Forces.**—As the forces in statics are supposed to be prevented by some kind of resistance from producing motion, the masses of the bodies acted upon seldom enter into the calculations. They generally occur as *weight*, as *pressing*, or as *pulling* forces, and can often be conveniently reckoned in gravitation units, i.e. in pounds-weight. We shall, however, very frequently speak of a force of 4 or 5 units without indicating the unit employed.

The complete specification of a force implies the knowledge of (1) its point of application, (2) its direction, (3) its magnitude.

All these elements can be represented geometrically. A point can be taken to represent the point of application of a force; a straight line drawn in a definite direction can represent the *sense* or direction of the force; and if a unit of length be taken to represent a unit of force, the number of units of length in the line will represent the number of units of force. Or, two forces P and Q will be represented by two lines, $A B$ and $C D$, when $P : Q :: A B : C D$. Care should be taken to distinguish the force $A B$ from the force $B A$, since these two forces, though equal in magnitude, act in opposite directions. As forces can be completely represented by straight lines, they may be treated as *vectors* or 'directed

quantities,' and the parallelogram and polygon laws, already considered in reference to velocities and accelerations, are applicable to them. It is useful, however, to consider independently the methods of compounding and resolving them.

§ 149. **Forces in the same Straight Line.**—If only two forces act upon a body it is clear that in order that they should produce no motion they must act (1) at the same point, (2) in opposite directions, and (3) they must be equal in magnitude.

If instead of two we have several forces acting at a point and in the same straight line, it is also evident that for equilibrium the tendency to motion in one direction must be counterbalanced by the tendency to motion in the opposite direction, or the sum of the forces in one direction must equal the sum of the forces in the opposite direction—i.e. the *algebraic* sum of the forces must vanish.

If X_1, X_2, X_3 . . . be the several forces acting in the same straight line and X denote their algebraical sum, then the condition of equilibrium is $X = 0$.

§ 150. **Forces in Equilibrium acting at a Point.**—Since the resultant is that force which produces the same effect as all the other forces

taken together, it is evident that when the forces produce equilibrium their joint effect in moving the body acted upon is zero, or the resultant vanishes. The resultant being determined, the conditions of equilibrium are the conditions that must hold good in order that this resultant may become zero. The solution of the equation $R = 0$, R being the numerical value of the resultant, will always determine the conditions of equilibrium when several forces act at a point.

If any number of forces acting at a point be in equilibrium, and one of them be removed, the resultant of all the rest is equal in magnitude, but opposite in direction, to the removed force; for, since the forces were originally in equilibrium, the removal of one force must destroy the equilibrium, since all the other forces served to counteract the effect of this one. But the single force which will counteract the effect of another force is one equal in magnitude and opposite in direction, and therefore a force equal in magnitude to the removed force, but opposite in direction, produces the same effect as all the remaining forces, or the removed force *reversed* is the resultant of the rest.

§ 151. It will be seen that when forces are in equilibrium motion may be produced either by the removal of one of the forces or by the application

of a new one. A balloon resting in mid-air will begin to rise if some ballast be thrown out, or to sink if gas be allowed to escape. From what we have shown in previous sections it is clear that if a body be acted upon by one force only, it will move, and with an accelerating speed; if a body, under the action of forces, be in equilibrium, or move uniformly in a straight line, two forces at least must co-operate.

§ 152. The resultant of any number of forces should be carefully distinguished from the force that keeps them in equilibrium, which is called the *equilibriant*. These two forces, the resultant and equilibriant, though equal in magnitude are opposite in direction. If AB is the *resultant* of any number of forces, BA is the *equilibriant*, and $AB + BA = 0$.

EXERCISES XVII

1. A body rests on a perfectly smooth table, and to one end of it is tied a string which is stretched by forces of 3, 5, and 7; and to the other end a string that is stretched by forces of 1, 8, and 2; what additional force is necessary to preserve equilibrium?
2. A string AD is suspended at A; at B, a point in the string, a mass of weight 3 oz. is attached, at C 4 oz., at D 5 oz.; find the force of the pull in each part of the string.
3. If a force of 13 units be represented by a line of $6\frac{1}{2}$ inches, what line would represent a force of 7 units?

4. The diagonal of an oblong is 5 centimetres, and one of its sides is 3 centimetres. Two forces acting in the same straight line are to one another as the sides of the oblong; what length of line would represent the force necessary to produce equilibrium?

XXI. *Composition of Forces acting at a Point, but not in the same Straight Line*

§ 153. We will, first, consider the case of *two* forces acting at a point but not in the same straight line.

It has been seen in Lesson IV. that if OA, OB represent two velocities or two accelerations, then OC, the diagonal of the parallelogram formed about OA and OB, represents the resultant velocity or acceleration. What is true of velocities or accelerations is true of the forces that would produce them, if only the forces act on the same mass, as is evident from the equation $F = Ma$. If, therefore, OA and OB represent two forces, OC represents the force equivalent to both, and CO the force that would keep them in equilibrium. This proposition, known as the Parallelogram of Forces, follows at once from Newton's second law of motion, and may be enunciated thus:—

If two forces acting at a point be represented in magnitude and direction by the adjacent sides

of a parallelogram, the resultant of these two forces will be represented in magnitude and direction by the diagonal of the parallelogram passing through this point.

§ 154. This proposition may be experimentally verified by passing a fine thread over two smooth pulleys fixed to a wall, as in the adjoining figure. From the ends of the thread hang two masses

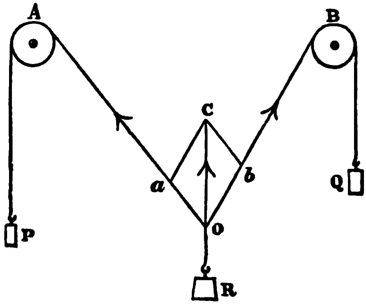

FIG. 55

weighing P and Q, and to some point in the thread a third mass of weight R is attached. When these weights are in equilibrium it will be found that a parallelogram may be constructed the sides and diagonal of which are nearly proportional to the three weights, and more nearly as friction at the pulleys is lessened. Thus, if P, Q, and R be in equilibrium at O, and $O\,C$ be taken to represent R, and $C\,a$, $C\,b$ be drawn parallel to $O\,B$, $O\,A$, then $O\,a$, $O\,b$, $C\,O$ will represent P, Q, and R respec-

tively. In other words, it will be found that $Oa : CO :: P : R$, and $Ob : CO :: Q : R$.

It should be noted that in this experiment R is the equilibriant of P and Q, and that OC represents R reversed, i.e. the resultant.

§ 155. **Resultant of several Forces acting at the same point in different directions.**—Let P_1, P_2, P_3, and P_4 be forces acting at a point O.

Let OA represent P_1
„ OB „ P_2
„ OC „ P_3
„ OD „ P_4.

Then by the foregoing proposition the resultant of P_1 and P_2 is a force represented by Oa; and

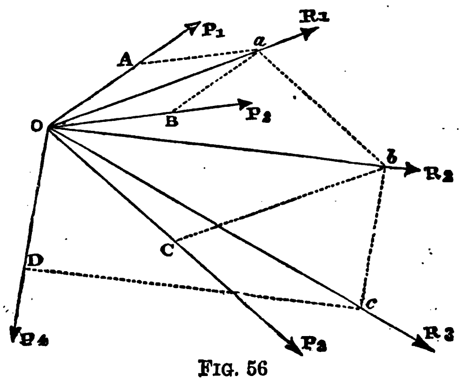

FIG. 56

if this be combined with OC, the new force represented by Ob will be the resultant of P_1,

P_2, and P_3. By combining this resultant with P_4, we obtain $O c$ the resultant of P_1, P_2, P_3, and P_4; and in the same way the resultant of any number of forces may be obtained.

The above method shows how the resultant of any number of forces may be *graphically* obtained. If the numerical value of the resultant has to be calculated, so many mathematical difficulties arise that in most cases it will be found convenient to employ another method, which will be considered later on.

With careful drawing the measured length of $O c$ gives the magnitude of the resultant nearly enough for most practical purposes, and many problems may be solved by drawing instead of by calculation.

§ 156. **Formula for the Resultant of two Forces acting at a point.**—Let $O A$, $O B$ represent

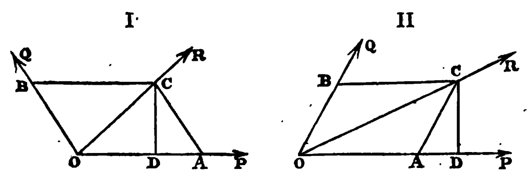

FIG. 57

P and Q, two forces acting at O. Complete the parallelogram $O A B C$. Then $O C$ represents the resultant. From C draw $C D$ perpendicular to

OA or OA produced. Then, since AC equals OB, AC represents Q. By Euclid II. 12 and 13 :

$$OC^2 = OA^2 + AC^2 \mp 2\,OA\,.\,AD,$$

or
$$R^2 = P^2 + Q^2 \mp 2\,P \times AD.$$

The upper sign — referring to fig. I., and the lower sign + to fig. II.

Thus R can be found in terms of P and Q, whenever the value of AD can be expressed in terms of AC. This, we shall see, is in several cases possible without the aid of trigonometry.

§ 157. **Special Cases of the General Formula.**— (1) Let the two forces act at right angles. Then, since $\angle BOA$ is a right angle, $AD = 0$ and $OC^2 = OA^2 + AC^2$, i.e. $R^2 = P^2 + Q^2$, or $R = \surd(P^2 + Q^2)$.

(2) Let the angle BOA be 30°. Then, in fig. 57, ii., the angle $CAD = 30°$ and $AD = AC\frac{\surd 3}{2}$, or $R^2 = P^2 + Q^2 + \surd 3\,PQ$.

(3) Let the angle BOA be 45°. Then the angle $CAD = 45°$ and $AD = AC\frac{1}{\surd 2}$,

or
$$R^2 = P^2 + Q^2 + \surd 2\,PQ.$$

In the same way, by reference to fig. 57, i., the values of R^2 may be found when the angle BOA is 120°, 135°, or 150°.

The results thus obtained may be arranged and remembered in the following form :—

If the angle between the forces be

	$30°,\ R^2 = P^2 + Q^2 + \sqrt{3} \,.\, P\,Q$
if	$45°,\ R^2 = P^2 + Q^2 + \sqrt{2} \,.\, P\,Q$
"	$60°,\ R^2 = P^2 + Q^2 + \quad P\,Q$
"	$90°,\ R^2 = P^2 + Q^2$
"	$120°,\ R^2 = P^2 + Q^2 - \quad P\,Q$
"	$135°,\ R^2 = P^2 + Q^2 - \sqrt{2} \,.\, P\,Q$
"	$150°,\ R^2 = P^2 + Q^2 - \sqrt{3} \,.\, P\,Q.$

It will be seen that the several values of the resultant lie between two extreme values when the angle $B\,O\,A$ is 0° and 180°. In these cases $R = P + Q$ and $P - Q$ respectively. Hence R^2 is always between $P^2 + Q^2 + \sqrt{4} \,.\, P\,Q$, and $P^2 + Q^2 - \sqrt{4} \,.\, P\,Q$.

§ 158. **Examples.**—(1) Single movable pulley with cords inclined. In considering pulleys we omitted

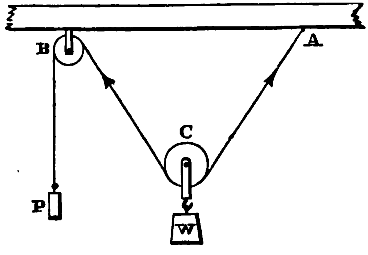

FIG. 58

cases in which the cords are inclined. Let C be a single movable pulley with cords inclined at any of the above-mentioned angles. Then it is clear that the weight W is the equilibriant, and is equal to the re-

sultant of the forces acting through the cords CA, CB, and these forces are equal to each other and to the pull P. If the angle between the cords be 60°,

$$W^2 = P^2 + P^2 + P \times P = 3P^2,$$

$$\therefore W = \sqrt{3}\,.\,P \text{ or } \frac{W}{P} = \sqrt{3}.$$

In the same way the ratio of W to P may be found when the cords are inclined at other angles.

(2) Two equal forces act along CB, BA, two sides of an equilateral triangle ABC; find their resultant. It will be seen that whilst the lines BC, BA are inclined to each other at an angle of 60° the lines CB, BA are inclined at an angle of 120°;

$\therefore R^2 = P^2 + Q^2 - PQ$, or if each of the forces equal P,

$$R^2 = P^2 + P^2 - P^2, \text{ or } R = P.$$

Hence, the resultant of two equal forces inclined at an angle of 120° is equal to either of them.

The directions in which two or more forces act are often indicated by saying that the forces act along the sides of a figure the angles of which are known. Thus, if $ABCDEF$ be a regular hexagon, two forces along AB, BC act at an angle of 60°, and two forces along AB, CB at an angle of 120°.

§ 159. If two forces are represented by AB and AC, the sides of a triangle, then their resultant will be represented by twice AD, where D bisects BC, the base of the triangle. For, if the parallelogram $ABEC$ be completed, AE represents the resultant of the two forces, and $AE = 2\,AD$, since the diagonals of a parallelogram bisect each

other. The line $A D$ is called the *median* line of the triangle $A B C$.

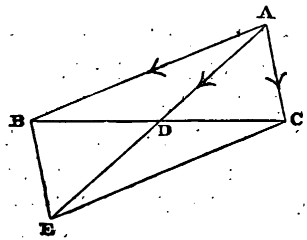

FIG. 59

§ 160. **The greater the angle between two forces, the less is their resultant.**—This proposition may be easily proved graphically, and follows from the consideration of § 156. It may, however, be independently demonstrated thus: Let P and

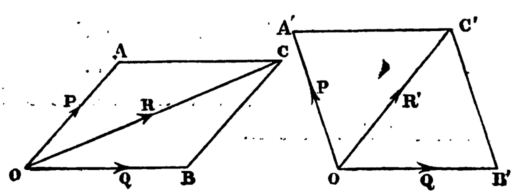

FIG. 60

Q be two forces, and let the angle between them be, first, $A\ O B$, and, secondly, $A'\ O\ B'$, and let the angle $A'\ O\ B'$ be greater than $A\ O\ B$. Then, since $O\ B = O B'$ and $B\ C = B'\ C'$, and the angle $O\ B\ C$ is greater than the angle $O\ B'\ C$, it follows that

OC is greater than OC'. Hence the resultant decreases as the angle between the forces increases.

§ 161. **The Resultant is always nearer to the greater force.**—The line OC is said to be nearer to OA than to OB when the angle it makes with OA is less than the angle it makes with OB. Let OA and OB represent two forces P and Q, of which P is the greater. Complete the parallelogram $OACB$. Then $OA = BC$; and, since BC is greater than

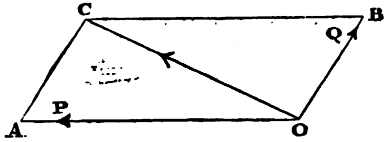

FIG. 61

OB, the angle COB is greater than the angle OCB. But the angle $OCB = \angle COA \therefore \angle COB$ is greater than $\angle COA$, and therefore OC is nearer to OA than to OB.

§ 162. **Resultant of three forces acting at a point, but not in the same plane.**—Let OA, OB, OC represent P_1, P_2, P_3, acting at O. Then the resultant of P_1 and P_3 is represented by OD, the diagonal of the parallelogram $OCDA$, and the resultant of this force and P_2 is represented by OE, the diagonal of the parallelogram $OBED$, and of

the parallelopiped of which the three lines OA, OB, OC are edges.

If the three forces act at right angles to one another,

$$R^2 = P_1^2 + P_2^2 + P_3^2.$$

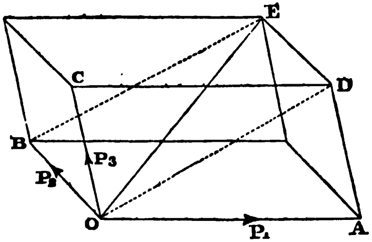

FIG. 62

EXERCISES XVIII

1. A particle is acted upon by two forces of 10 units and 12 units, one of which is inclined at an angle of 80° to the vertical, and the other at an angle of 40° to the vertical and on the other side of it; find the magnitude of the single force which would produce the same effect as these two conjointly.
2. Find the resultant of two forces acting at an angle of 45°, one of which is twice the other.
3. Two forces, one of which is three times the other, act along the adjacent sides of a square; find the resultant.
4. Three equal forces act along the sides AB, BC, DC of a square $ABCD$; find the resultant.
5. Forces of 8 lbs. wt. and 10 lbs. wt. act at an angle of 60°; find their resultant.
6. The resultant of two forces that act at right angles to one another is 145 lbs. wt., and one of the forces is 144 lbs. wt.; find the other.

7. Two forces whose magnitudes are as 3 to 4 act at a point in directions at right angles, and produce a resultant of 2; find the forces.
8. The directions of two forces acting at a point are inclined to each other (1) at an angle of 60°, (2) at an angle of 120°, and the respective resultants are in the ratio $\sqrt{7} : \sqrt{3}$; compare the magnitude of the forces.
9. Forces of 4 lbs. wt. and 5 lbs. wt. act along the sides AB, BC of an equilateral triangle; find the resultant.
10. A boat is moored in a stream by two ropes, one fastened to either bank, and the ropes make an angle of 90° with each other. The force of the stream is equal to 500, and the pull on one of the strings is 300 lbs. weight; find the pull on the other.
11. Two forces of 4 and $3\sqrt{2}$ act at an angle of 45°, and a third force of $\sqrt{42}$ acts at right angles to their plane at the same point; find their resultant.
12. Three equal rods are joined at a point, and at right angles to one another, and a load of $4\sqrt{3}$ lbs. hangs from their point of intersection; find the pressing force transmitted through each rod.
13. Four equal forces act at a point; the first is at right angles to the second, the third is at right angles to the resultant of the first two, and the fourth is at right angles to the resultant of the other three; find the resultant of all four.
14. Three smooth posts are placed in the ground so as to form an equilateral triangle, and an elastic ring is stretched round them, the pull of which is 6 poundals; find the pressing force on each post.
15. Six smooth posts are fixed in the ground so as to form a regular hexagon, and a cord is passed twice round them, and pulled with a force of 100 units; find the magnitude and direction of resultant force on each post.
16. Two forces of the magnitude 5 and 11 act at angles of 60°, 90°, 120° respectively; compare their resultants in the three cases.

XXII. *Geometrical Condition of Equilibrium when two or more Forces act at a Point*

§ 163. **Triangle of Forces.**—If $O A$ and $O B$ be two forces acting at O, and the parallelogram $O A B C$ be completed, $O C$ will represent the resultant of these two forces, and $O C$ reversed or $C O$ will keep $O A$ and $O B$ in equilibrium. Since

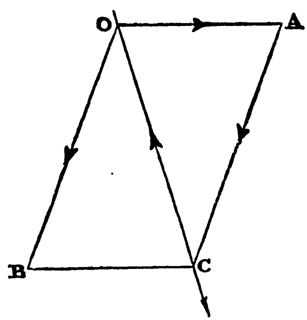

FIG. 63

$A C$ equals $O B$, the three forces represented by $O A$, $A C$, and $C O$ will be in equilibrium, if they act at a point. Conversely, if three forces acting at a point are in equilibrium, since any one reversed equals the resultant of the others, the three forces can be represented by the sides of a triangle *taken in order.*

If a triangle be drawn anywhere else, with its sides respectively equal to the sides of the triangle

O A C, it can be made to coincide with it, and will therefore represent the *magnitude* of the three forces in equilibrium. In this case the triangle does not represent the directions, but only the *relative directions* of the forces.

The proposition known as the *triangle of forces* is generally enunciated thus :—

'If three forces acting at a point be in equilibrium, and if any triangle be drawn, the sides of which are respectively parallel or perpendicular to their directions, the forces will be to one another as the sides of the triangle; and, conversely, if the three forces are to one another as the sides of the triangle, they will be in equilibrium.'

§ 164. The proposition thus enunciated may be stated more generally, for the sides of a triangle

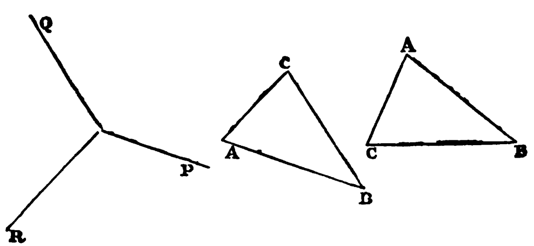

FIG. 64

may represent the magnitudes of the forces, though they are neither parallel nor perpendicular to their directions. Nothing more is necessary than that

the triangle shall be such as is capable of being taken up and twisted into parallelism with the sides. In other words, if three forces acting at a point be in equilibrium, and there be a triangle which can be so moved that its sides shall become respectively parallel to the directions of the forces, then the sides of this triangle, *taken in order*, will represent the forces; and, conversely, if the three forces can be represented by the sides of such a triangle, they will be in equilibrium.

Thus, if P, Q, and R be three forces in equilibrium, and $A\,B\,C$ a triangle the sides of which can be brought into parallelism with the directions of the forces, then

$$P : Q : R :: AB : BC : CA.$$

It is necessary to observe that, if the sides of the triangle be not taken in order, they will represent two forces and their resultant. Thus, AB, BC, and AC represent P, Q, and their resultant R.

This proposition—viz. the triangle of forces—is the statical supplement to the parallelogram of forces, as it gives the geometrical conditions of equilibrium when three forces act at a point.

§ 165. **Examples.**—(1) Equilibrium on the inclined plane. When a heavy body is supported on an inclined plane, whether by a force parallel to the plane or parallel to the base, the relation between the forces in

equilibrium can be deduced from the triangle of forces.

First. Let the force P act parallel to the plane AB. Let W be the weight of the body, and R the normal reaction of the plane. Then P, W, and R are supposed to be in equilibrium. Produce AC to B' (fig. 65) and make $AB' = AB$. From B' draw $B'C'$ perpendicular to AB. Then the triangle $AB'C'$ is in every respect equal to the triangle ABC, and its sides are respectively parallel to the directions of the three forces.

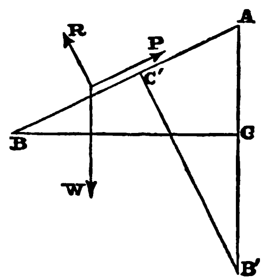

FIG. 65

Therefore the sides of the triangle $AB'C'$ represent the forces P, W, R. But the triangle $AB'C'$ is only a new position of the triangle ABC, which the original triangle can be made to assume.

$$\therefore P : W : R :: C'A : AB' : B'C$$
$$:: CA : AB : BC,$$

or
$$\frac{P}{W} = \frac{CA}{AB} = \frac{h}{l} \text{ and } \frac{P}{R} = \frac{CA}{BC} = \frac{h}{b}.$$

Secondly. Let the force P act parallel to the base of the plane. Then the triangle ABC can be turned

rcund the point C into the position $A'B'C$; in which case CA' is parallel to P, $A'B'$ to R, and $B'C$ to W;

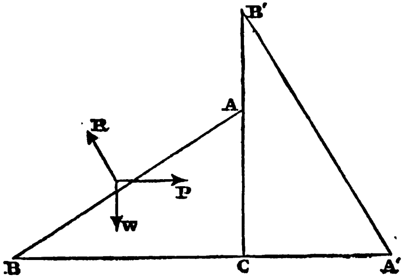

FIG. 66

$$\therefore P : R : W :: CA : AB : BC,$$

or $$\frac{P}{R}=\frac{CA}{AB}=\frac{h}{l} \text{ and } \frac{P}{W}=\frac{CA}{BC}=\frac{h}{b}.$$

(2) Two cords are each tied to a body weighing 25 oz. at C, and from their other extremities hang two bodies weighing P and Q. The cords pass over the smooth

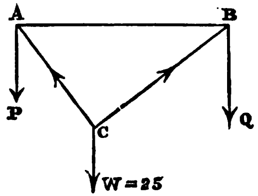

FIG. 67

pegs A and B, in the same horizontal line. If AC be 8 inches, and BC be 4 inches, and AB be 5 inches, find P and Q

Here the three forces acting at C are $W = 25$ oz. and P and Q the pulls in the strings CA, CB. Since $5^2 = 4^2 + 3^2$, it follows that the angle ACB is a right angle (Euc. I. 48). Thus AB, BC, and CA are respectively perpendicular to the directions of W, of the pull in CA, and of the pull in CB, and, therefore, the triangle ABC represents in magnitude the three forces in equilibrium, and

$$W : P : Q :: AB : BC : CA$$
$$:: 5 : 4 : 3;$$
$$\therefore P = \frac{4}{5} W = 20 \text{ oz., and } Q = \frac{3}{5} W = 15 \text{ oz.}$$

(3) Required the least horizontal force necessary to draw a heavy wheel over an obstacle the height of which is h, situated on the horizontal plane on which the wheel rests.

Let W be the weight of the wheel, r its radius, and let P be the force which is just able to lift the wheel

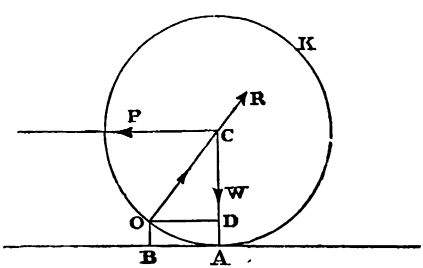

FIG. 68

over the obstacle. Let OAK be the wheel just on the point of rising from the ground, and let OB be the obstacle over which it is to be drawn. Then, as the

wheel is on the point of moving, the three forces in equilibrium are P and W at C, the centre of the wheel, and the reaction R of the obstacle $O\,B$ acting at O, at right angles to the circumference of the wheel, and, therefore, likewise passing through C. Now, it will be seen that the sides of the triangle $O\,C\,D$ are respectively parallel to the directions of the forces, and are, therefore, proportional to them. Hence

$$P : R : W :: D\,O : O\,C : C\,D$$

or
$$\frac{P}{W} = \frac{D\,O}{C\,D} = \frac{\sqrt{(r^2 - C\,D^2)}}{C\,D} = \frac{\sqrt{r^2 - (r-h)^2}}{r-h}$$

$$\therefore\ P = \frac{\sqrt{(2\,hr - h^2)}}{r-h} \cdot W.$$

§ 166. **Polygon of Forces.**—If any number of forces P_1, P_2, P_3, P_4, P_5 . . . acting at a point

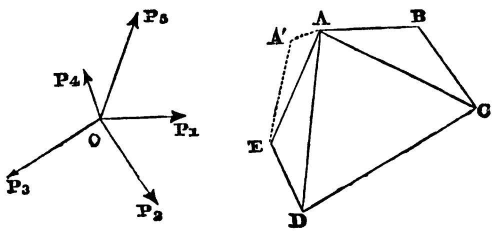

FIG. 69

O be represented by $A\,B$, $B\,C$, $C\,D$, $D\,E$, $E\,A$, the sides of a closed polygon taken in order, the forces shall be in equilibrium ; and if the forces be in equilibrium, they shall be capable of being represented by the sides of a polygon.

Join $A\,C$, $A\,D$. Then $A\,C$ represents the resultant of $A\,B$ and $B\,C$; and $A\,D$ represents the resultant of $A\,C$ and $C\,D$, i.e. of $A\,B$, $B\,C$, and $C\,D$; and $A\,E$ represents the resultant of $A\,D$ and $D\,E$, i.e. of $A\,B$, $B\,C$, $C\,D$, and $D\,E$; therefore $E\,A$, together with $A\,B$, $B\,C$, $C\,D$, and $D\,E$, produces equilibrium. Hence P_1, P_2, P_3, P_4, P_5 represented by the sides of the polygon, taken in order, are in equilibrium. The converse may be similarly proved.

If the lines representing the forces do not form a closed polygon—that is, if $E\,A'$, drawn parallel to P_5 and representing it in magnitude, does not meet $A\,B$ in A—the forces are not in equilibrium. In this case the line joining the points $A\,A'$ represents the resultant of the forces, and $A'\,A$ would keep them in equilibrium. In fig. 69 it will be seen that $O\,A\,a\,b\,c$ is an open polygon, the sides of which represent P_1, P_2, P_3, and P_4, and that $O\,c$, the line closing it, represents their resultant.

EXERCISES XIX

1. Three forces whose magnitudes are as 3 : 2 : 7 act at a point; can they be in equilibrium?
2. Three equal forces act at a point and are in equilibrium; what triangle will represent them?
3. Two forces acting at a point are represented by $A\,B$ the base, and $C\,A$ one of the sides of an isosceles triangle; find the resultant.

4. A cord fixed at its extremities to two points in the same horizontal line supports a ring weighing 10 oz.; the two parts of the cord contain an angle of 60°; find the pull of the cord.

5. Three forces of 20 lbs., 40 lbs., 50 lbs. wt. act at the same point, and make angles of 30°, 60°, and 90° respectively with a given straight line; determine their resultant.

6. A body weighing 48 lbs. is supported by two strings which are respectively 3 feet and 4 feet long, and are fastened to points in the same horizontal line at such distances apart that the strings make right angles with each other; find the pull in each.

7. Show that three forces acting at a point, but not in the same plane, cannot be in equilibrium.

8. $ABCDEF$ is a regular hexagon, and forces represented by the lines AB, AC, AD, AE, AF act at A; find their resultant.

9. Two forces acting at a point are represented by the semidiagonals AO, OD of a parallelogram $ADBC$; show which of the sides represents their resultant.

10. Three forces acting at a point are represented by three adjacent sides of a regular hexagon taken in order; find their resultant.

11. A boat is moored to two points on opposite banks, so that the line joining them is perpendicular to the direction of the stream and 15 feet in length. The two ropes make a right angle with each other, and one of them is 12 feet long, and the force of the stream is equivalent to 20 lbs. wt.; find the pull in each rope.

12. Three flexible strings A, B, and C are fastened to a small smooth ring; A and C pass without friction over two fixed pulleys; what weight must be applied to B in order that the forces acting on the ring may be in equilibrium in each of the following cases?

(*a*) Weight at $A=9$ oz., weight at $C=9$ oz.; angle between the strings A and $C=120°$.
(*b*) Weight at $A=9$ oz., weight at $C=9$ oz.; angle between the strings $=180°$.
(*c*) Weight at $A=9$ oz., weight at $C=18$ oz.; angle between the strings $=150°$.
(*d*) Weight at $A=9$ oz., weight at $C=12$ oz.; angle between the strings $=90°$.

XXIII. *Resolution of Forces—Analytical Conditions of Equilibrium when any number of Forces act at a Point*

§ 167. When a single force is replaced by other forces, which together produce the same effect, it is said to be resolved, and the several resolved parts are called *components*.

If a heavy body be pulled along a road by a cord inclined to it at an angle, it is evident that the whole of the force employed is not used in drawing the body *along* the road, but that it also tends to lift the body *off* the road. That force which produces in a given direction the same effect as another force acting in a different direction is called the resolved part of the other in that direction.

§ 168. It is evident from the parallelogram of forces that any single force can have two components in any directions whatever, since the same

straight line may be the diagonal of any number of different parallelograms. The most convenient components into which a force can be resolved are those the directions of which are at right angles to each other. Thus, if P be a force acting at O, and represented by OA, then, since OA is the diagonal of the rectangle $OABC$, the two forces represented by OB and OC produce the same

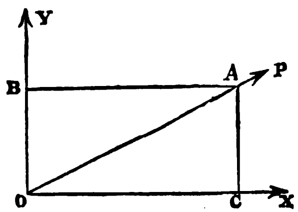

FIG. 70

effect as P; and if we call X the force acting along OC, and Y the force acting along OB

$$P^2 = X^2 + Y^2.$$

This method of resolution is the simplest, because no part of X aids or opposes Y. Neither component has any part in the other.

§ 169. **The projection of a force in any line represents the resolved part of the force along that line.**—From the preceding it follows that the resolved part of OA in the direction Ox is represented by the line intercepted between O and the foot of the perpendicular let fall from A on the line

$O x$; and if $O' x'$ be drawn parallel to $O x$, and $O O'$, $A B'$ be drawn perpendicular to $O' x'$, then $O' B' = O B$. Hence $O' B'$ represents, in magnitude and direction, the resolved part of $O A$ in the direction $O' x'$. But $O' B'$ is called the *projection* of $O A$ on the line $O' x'$. Therefore the resolved

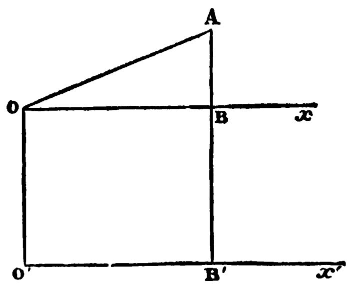

FIG. 71

part of a force in any line equals the projection of that force on the given line.

§ 170. Forces may very often be compounded by this method of resolution with greater facility than by the parallelogram or polygon of forces. For, if each of the several forces acting at a point be resolved along the same two lines at right angles, the sum of their components in either direction can at once be found, and the system is thus reduced to two forces at right angles.

§ 171. **To find the resultant of any number of forces acting at a point in different directions**

and in the same plane.—Take three coplanar[1] and concurrent forces P_1, P_2, P_3, represented by $O\,A$, $O\,B$, and $O\,C$ respectively. Take $x\,x'$, $y\,y'$, two straight lines at right angles through O. Project $O\,A$, $O\,B$, and $O\,C$ on each of these lines. Then the three forces may be replaced by $O\,L$ and $O\,L'$, by $O\,M$ and $O\,M'$, by $O\,N$ and $O\,N'$. But $O\,L$, $O\,M$, and $O\,N$ act in the same straight line, and therefore their resultant X is their algebraic

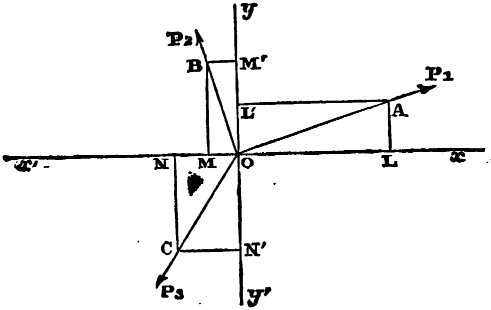

FIG. 72

sum. Also, $O\,L'$, $O\,M'$, $O\,N'$ act in one straight line, and have a resultant Y equal to their algebraic sum. Thus the original forces P_1, P_2, and P_3 are equivalent to X and Y at right angles, where

[1] Three or more forces are said to be *coplanar* when they all act in the same plane, and *concurrent* when their directions all pass through the same point. All the forces are supposed to be *coplanar* unless otherwise described.

$$X = OL - MO - NO,$$

and $$Y = OL' + OM' - N'O,$$

and if R be the resultant of X and Y, i.e. of all the forces,

$$R^2 = X^2 + Y^2.$$

In the same way the resultant of any number of forces may be obtained. Hence, if any number of forces act at a point, and two lines be drawn through this point at right angles to each other, and if X and Y be the algebraic sums of the components of these forces along these lines respectively, then $R^2 = X^2 + Y^2$, where R is the numerical value of the resultant of all the forces.

The direction of the resultant is the angle made with Ox by that diagonal of the rectangle whose sides are X and Y which passes through O.

§ 172. **Examples.**—(1) Three forces of 4, 5, and 6 units act at a point. The angle between the first and second is 45°, and between the second and third 75° Find their resultant.

Take OA, OB, and OC to represent the three forces. It is generally preferable to take for the line xOx' either the line of action of one of the forces or some other line which makes convenient angles with the forces. In this case take xOx' along OA, and draw yOy' at right angles to it through O. Project the forces along Ox and Oy. Then, as before, we have

$$X = OA + OM - NO$$

$$= 4 + \frac{5}{\sqrt{2}} - 3 = 1 + \frac{5}{\sqrt{2}};$$

$$Y = OM' + ON' = \frac{5}{\sqrt{2}} + 3\sqrt{3},$$

and

$$R^2 = \left(1 + \frac{5}{\sqrt{2}}\right)^2 + \left(\frac{5}{\sqrt{2}} + 3\sqrt{3}\right)^2$$

$$= \tfrac{1}{2}\left\{(\sqrt{2} + 5)^2 + (5 + 3\sqrt{6})^2\right\}$$

$$= 53 + 5\sqrt{2} + 15\sqrt{6};$$

$$\therefore R = 10 \text{ nearly}.$$

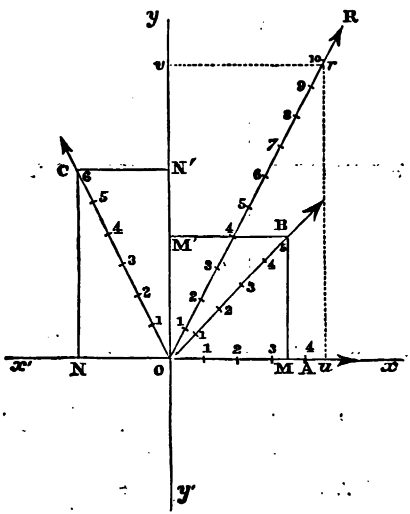

FIG. 73

If Ou be taken equal to $OA + OM - NO$, and Ov be taken equal to $OM' + ON'$; and if ur, vr be drawn parallel to Oy and Ox respectively, Or will

represent the resultant, and this line will be found to measure a little less than ten units.

(2) To find the resultant of equal forces when the angle between them is 15°.—Let the line $x\,O\,x'$ be so taken that it makes an angle of 30° with one force and 45° with the other, in which case the angle between the forces will be the difference of 45° and 30°, that is, 15°. Draw $O\,y$ at right angles to $O\,x$.

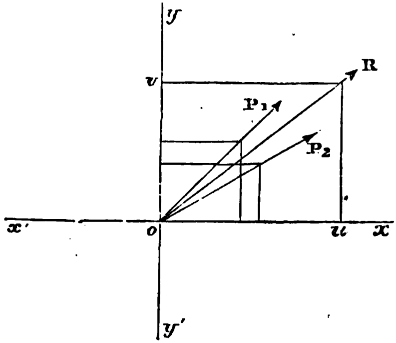

FIG. 74

The projections of P_1 are $\dfrac{P_1}{\sqrt{2}}$ and $\dfrac{P_1}{\sqrt{2}}$.

" " P_2 are $\dfrac{P_2\sqrt{3}}{2}$ and $\dfrac{P_2}{2}$.

But $P_1 = P_2$,

$$\therefore X = ou = P\left(\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\right);$$

and $$Y = ov = P\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right),$$

$$\therefore R^2 = X^2 + Y^2 = \frac{P^2}{2}(4 + \sqrt{2} + \sqrt{6}),$$

$$\therefore R = P(1{\cdot}98\cdots)$$

§ 173. **Conditions of Equilibrium when any number of Coplanar Forces act at a Point.**—If the forces are in equilibrium it is evident that the resultant must be zero; and since by resolving we have (§ 171) $R^2 = X^2 + Y^2$, it follows that $X^2 + Y^2 = 0$ when the forces are in equilibrium, and consequently $X = 0$ and $Y = 0$, since X^2 cannot equal $-Y^2$. Hence, *if any number of forces acting at a point are in equilibrium the algebraic sums of the components of these forces, along any two straight lines drawn at right angles to each other through this point, must separately vanish.*

The conditions of equilibrium thus found are called the *analytical* conditions, because the forces are analysed into their components. Remembering what X and Y represent, these two conditions will be indicated by the equations

$$(1)\ X = 0,$$
$$(2)\ Y = 0.$$

EXERCISES XX

1. Two equal forces acting at a point make angles of 30° and 45° with opposite sides of the same straight line; find their resultant.
2. Two forces of 6 lbs. wt. and 10 lbs. wt. acting at a point include an angle of 105°; find the resultant.
3. Two forces of 8 lbs. wt. and 10 lbs. wt. act at right angles; find the resultant by resolving each along two directions at right angles, one of which makes an angle of 30° with the direction of one of the forces.

4. Three forces of 4, 5, and 6 units act at a point and include angles of 60° and 75°; find their resultant.

5. Three equal forces act at a point, and the angle included between the first and second is 30°, between the second and third, 105°; find the resultant.

6. Five forces of 5, 2, 4, 3, and 6 units act at a point and include angles of 45°, 75°, 60°, and 90°; find the magnitude of the force which will keep them in equilibrium.

7. Show why the traces of a horse ought always to be parallel to the road along which he is pulling.

8. A man pulls a heavy body by means of a rope along a road with a force of 20 lbs. wt. The rope makes an angle of 30° with the road. Find the force he would need to apply parallel to the road. Does the inclination of the road alter the answer?

9. Find the magnitude of the force which will keep in equilibrium two forces of 2 units and 4 units, (1) at an angle of 15°, (2) at an angle of 75°.

10. The resultant of two forces acting at a point is 4 and the angle between them is 150°. One of the components is 4; find the other.

11. Let $A\ B\ C$ be a triangle, and lines be drawn from A, B, C to the middle points of the opposite sides. If three forces acting from A, B, C respectively be represented by these three lines the forces will be in equilibrium.

12. Three forces act at a point, and include angles of 90° and 45°. The first two forces are each equal to $2\ P$, and the resultant of them all is $\sqrt{10}\ P$; find the third force.

13. Indicate the forces that maintain a kite in equilibrium.

14. Find the resultant of three forces, the least of which is 10, which are represented by and act along $O\ A$, $O\ B$, $O\ C$, two sides and the diagonal of an oblong whose area is 60 square inches, and shorter side 5 inches.

15. A man and a boy pull a heavy body by ropes inclined to the horizon at angles of 60° and 30° with forces of 80 lbs. wt. and 100 lbs. wt. The angle between the two vertical planes of the cords is 30°; find the single horizontal force that would produce the same effect.
16. The resultant of two equal forces is 56, and the included angle is 15°; find the forces.
17. A body weighing 10 lbs. is supported by two strings, one of which makes an angle of 30° with the vertical, the other 45°; find the pull in each string.
18. Three forces each equal to 10 act at the same point; the first makes an angle of 30° with the second, and the second an angle of 60° with the third; calculate the magnitude of the single force which with them will produce equilibrium.

XXIV. *Forces not meeting at a Point—Parallel Forces*

§ 174. Parallel forces are those which act at different points of a body and in directions parallel to one another. If they act in the same direction they are called *like* forces; if in opposite directions, *unlike*. A pair of horses harnessed to a tram-car is an example of two like parallel forces.

§ 175. **Resultant of two Parallel Forces.**—If two forces are parallel and act in the same direction, it is evident that their joint effect is the sum of their separate effects; and if they act in opposite directions it is equal to their difference.

The magnitude of the resultant of parallel forces is, therefore, the same as of forces in the same straight line, i.e. is equal to their algebraic sum.

It has been shown (§ 161) that when two forces act at a point their resultant is nearer to the greater force. If the forces act in like directions, from or towards the same parts, the resultant will be found to act *between* them, as in fig. 75, i.; if in

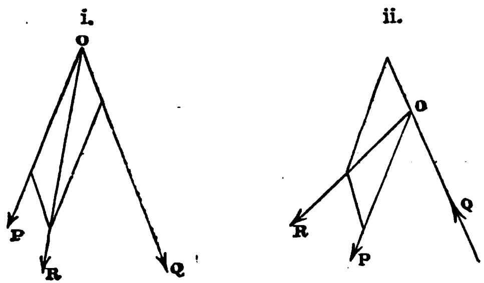

FIG. 75

unlike directions, it will be found to act *outside* both forces, as shown in fig. 75, ii. This proposition is found to be equally true when the forces are parallel, in which case the point O is infinitely distant. In fact, parallel forces may be regarded as a particular case of *concurrent* forces, the point of meeting O being infinitely far away. But in this case the nearness of the resultant to either of the forces can be more easily calculated than

when the forces are inclined at a finite angle, since it can be exactly measured by the perpendicular distance of the direction of the resultant from that of each of the forces. It is thus found that in parallel forces the resultant is as much nearer to the greater force as that force is greater than the lesser force, and acts between like and outside unlike parallel forces.

If P and Q be two parallel forces, *like* in fig. 76, i., and *unlike* in fig. 76, ii., the resultant R

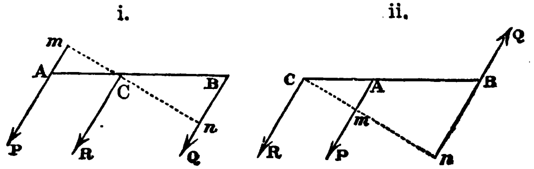

FIG. 76

will act between P and Q in fig. i., and outside them in fig. ii.; but in both cases, if P is greater than Q, R will be as much nearer to P as P is greater than Q; i.e. R's perpendicular distance from P will be as many times less than R's perpendicular distance from Q as P is greater than Q. Hence, if $m\,n$ or $m\,n$ produced be perpendicular to their directions,

$$P : Q :: C\,n : C\,m$$

and $\therefore P : Q :: C\,B : C\,A$ (by similarity of triangles).

Hence the magnitude and position of the resultant is determined by the equations

$$R = P \pm Q$$

and
$$P \times C\,A = Q \times C\,B.$$

§ 176. **Proof.** — Having thus explained the position of the resultant, we now proceed to prove the above proposition.

Case I. *Like Forces.*—Suppose the two forces to be acting at A and B, and to be represented by $A\,a$ and $B\,b$ (fig. 77). At A apply any force X, represented by $A\,x$, in the direction $B\,A$, and at B an equal force X', represented by $B\,x'$, in the direction $A\,B$. These forces will produce no effect, since they are equal and opposite.

Complete the parallelogram, as shown, and draw the diagonals $A\,s$, $B\,s'$. Then $A\,s$ represents S, the resultant of P and X; and $B\,s'$ represents S', the resultant of Q and X'. These two pairs of forces can, therefore, be replaced by S and S'. If the lines $A\,s$, $B\,s'$ be produced, they will meet at some point O, since the angles $x\,A\,s$, $x'\,B\,s'$ are together less than two right angles; and the forces S and S' may therefore be transferred to O.

Draw $O\,C$ parallel to $A\,a$ or $B\,b$, and resolve S into its two components, one along $O\,C$ and the other parallel to $A\,B$, and let S' be similarly resolved. Then we have X and X' acting at O in opposite directions, and counteracting one

another; and P and Q acting along OC. Hence $P + Q$ at C produces the same effect as P at A and Q at B, and therefore the resultant of P and Q acts at C parallel to their direction. Let R be the resultant; then $R = P + Q$.

Now, the triangle of OCA has its sides parallel to the directions of P, X, and S, and therefore,

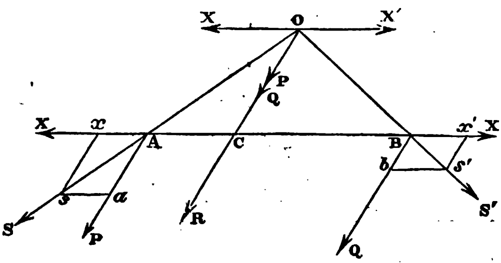

FIG. 77

since S is the resultant of P and X, we have by the triangle of forces,

$$\frac{P}{X} = \frac{OC}{CA}.$$

Similarly $$\frac{X'}{Q} = \frac{CB}{OC};$$

$$\therefore \quad \frac{P}{Q} = \frac{CB}{CA}, \text{ since } X = X';$$

or $$P \times CA = Q \times CB.$$

CASE II. *Unlike Forces.*—Let P and Q be two unlike forces acting at A and B, and represented

by $A\,a$ and $B\,b$ respectively, and let Q be greater than P. At A and B apply two equal forces X and X' in the directions $B\,A$ and $A\,B$. Then, completing the parallelograms as before, $A\,s$ will represent S, the resultant of P and X, and $B\,s'$ will represent S', the resultant of Q and X'. Since $B\,b$ is greater than $A\,a$, it is evident that the lines $s\,A$ and $B\,s'$ are not parallel, but will meet if produced at some point towards S'. Let them meet at O.

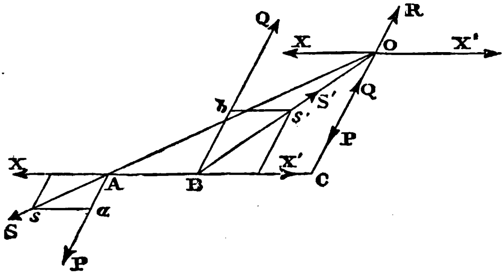

FIG. 78

Then S and S', being supposed transferred to O, may be resolved into their component parts, viz. X and X' parallel to $A\,B$ destroying one another, and P acting along $O\,C$ and Q in the direction of $C\,O$. Hence P at A and Q at B are equivalent to a force R equal to $Q - P$ in the direction $C\,O$. Therefore the resultant of P and Q is equal to $Q - P$, and acts parallel to the original forces at the point C in the direction of the greater force.

Also, since the sides of the triangle $A\,C\,O$ are parallel to the directions of P, X, and S, and the sides of the triangle $B\,C\,O$ are parallel to the directions of Q, X', and S', we have, as before,

$$\frac{P}{X} = \frac{O\,C}{C\,A} \text{ and } \frac{X'}{Q} = \frac{B\,C}{C\,O},$$

and $\therefore$ $$\frac{P}{Q} = \frac{B\,C}{C\,A}, \text{ since } X = X',$$

or $$P \times C\,A = Q \times B\,C.$$

§ 177. **Experimental Verification.**—The above propositions may be experimentally verified in the following way :—

Let $X\,X'$ be a uniform rod suspended at its middle point C. Let two masses weighing P and Q

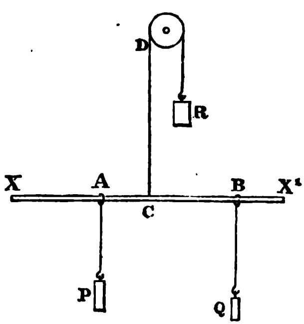

FIG. 79

respectively hang from any two points A and B (fig. 79), and let a mass of weight R equal to their

sum be attached to the end of a thin cord that passes without friction over the pulley D. Then if the two masses be moved till there is equilibrium, and if the distance $C A$ be equal to p, and the distance $C B$ equal q, it will be found that $P \times p = Q \times q$.

Again, let the mass of P (fig. 80) hang from A, and that of Q be fastened to the end of the string that passes over the pulley. Then P and Q are two *unlike* forces, and if Q be greater than P,

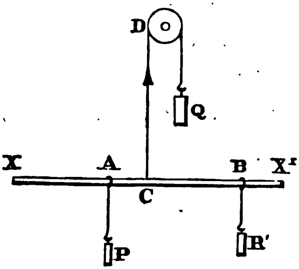

FIG. 80

it will be found necessary to hang a body weighing R' from some point B to maintain equilibrium, and if $R' = Q - P$, and the distance $A B = p$, and the distance $C B = q$, it will be found that $P \times p = Q \times q$, as before.

It should be carefully noted that the forces R and R' in the above experiments are *not* the *resultants* of P and Q, but, being the equilibriating forces, are equal and opposite to these resultants.

For the success of the experiment the weight of the rod $X X'$ should be small compared with the weights P, Q, R, or, better still, should be first counterbalanced by a small additional weight attached to the cord that passes over the pulley.

§ 178. **Examples.**—(1) If $P = 4$ and $Q = 5$ be two like parallel forces, and the distance between their points of application be 12 inches, find the position of the resultant.

$$\begin{aligned} &\text{Let } x \text{ equal } R\text{'s distance from } P, \\ &\text{then } 12 - x \text{ equals } \quad ,, \quad ,, \quad ,, \quad Q, \\ &\therefore 4 \times x = 5\,(12 - x) = 60 - 5\,x, \\ &\therefore x = 6\tfrac{2}{3} \text{ and } 12 - x = 5\tfrac{1}{3}. \end{aligned}$$

(2) If the resultant of two unlike parallel forces is 12, and acts at a distance of 5 inches from the greater and 7 inches from the lesser force, find the forces.

Let Q be greater than P, then $R = Q - P = 12$

and $\quad \therefore Q = 12 + P,$

also $\quad 5 \times Q = 7 \times P;$

$$\therefore 5 \times (12 + P) = 7 \times P,$$

$$\therefore P = 30 \text{ lbs. and } Q = 42.$$

(3) The resultant of two like forces is 10, and is twice as near to P as to Q; find the forces.

$$P + Q = 10 \therefore Q = 10 - P.$$

Let x be P's distance from R.

$$\begin{aligned} \text{Then } P . x &= (10 - P) . 2\,x, \\ \therefore P &= 20 - 2\,P, \\ \therefore P &= 6\tfrac{2}{3} \text{ and } Q = 3\tfrac{1}{3}. \end{aligned}$$

§ 179. **To find the Position of the Resultant of a number of Parallel Forces acting at different**

points of a rigid body.—Let P_1, P_2, P_3 . . . be parallel forces acting at the points A, B, C . . . which are all rigidly connected, but do not necessarily all lie in one plane. Then the resultant of P_1 and P_2 is R_1, $= P_1 + P_2$, and acts at a point O_1, such that $P_1 \times AO_1 = P_2 \times BO_1$. Similarly R_2, the resultant of R_1 and P_3, acts at a point O_2, such that $R_1 \times O_1 O_2 = P_3 \times CO_2$,

or $$(P_1 + P_2) \times O_1 O_2 = P_3 \times CO_2$$

and $$R_2 = P_1 + P_2 + P_3.$$

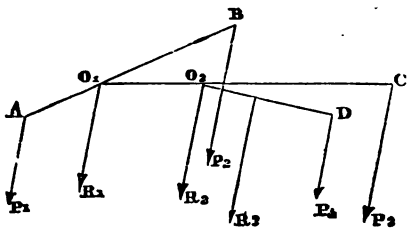

FIG. 81

In the same way, by combining this resultant with a new force, and the resultant of these with another force, the resultant of any number of parallel forces may be obtained.

§ 180. **Centre of Parallel Forces.**—The point at which the resultant of any number of parallel forces acts is called the *centre* of the forces. It is evident from the foregoing investigation that the position of the centre of any number of parallel

forces is independent of the directions of the forces, and depends only on their *points of application* and their respective *magnitudes.*

When the forces are all in one plane the position of this point may be easily found in most cases by the following method :—

If P_1 and P_2 be the forces, and O any point ; and if through O any line be drawn cutting their directions in the points A and B and their resultant

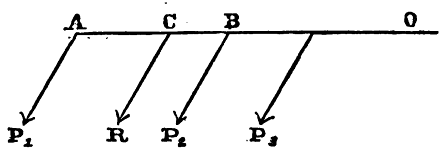

FIG. 82

at C, then, since $P_1 \times A\,C = P_2 \times B\,C$, it follows that $P_1 \times (O\,A - O\,C) = P_2 \times (O\,C - O\,B)$

and $P_1 \times O\,A + P_2 \times O\,B = (P_1 + P_2) \times O\,C$,

$$\therefore\ P_1 \times O\,A + P_2 \times O\,B = R \times O\,C.$$

Similarly by combining R with P_3, a third force in the same plane as P_1, P_2, and R, and the resultant of R and P_3 with a new force, and so on, we can prove that if $P_1, P_2, P_3 \ldots$ be any number of parallel forces, $x_1, x_2, x_3 \ldots$ their respective distances from a point O along any straight line drawn through that point, and x the distance of their resultant from the same point, then

$$Rx = P_1 x_1 + P_2 x_2 + P_3 x_3 + \ldots$$

and $$\therefore x = \frac{P_1 x_1 + P_2 x_2 + P_3 x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}$$

§ 181. **Equilibrium of Parallel Forces.**—If P_1, P_2, P_3 . . . be co-planar parallel forces in equilibrium, some acting in one direction and some in the opposite direction, their resultant equals zero, and the conditions of equilibrium are therefore given by the equations,

(i.) $P_1 + P_2 + P_3 + \ldots = 0$,
(ii.) $P_1 x_1 + P_2 x_2 + P_3 x_3 + \ldots = 0$.

In these equations forces pulling in opposite directions are supposed to have opposite signs; and if the point O, through which the line of reference passes, and which may be taken anywhere, lies between the forces, some being on one side of it and some on the other, a further distinction of sign is necessary, and all the *distances* on one side of O must be accounted positive, and those on the other side negative. Bearing in mind these distinctions of sign, the conditions of equilibrium may be thus stated : *Parallel forces in the same plane are in equilibrium, when* (1) *the algebraic sum of the forces equals zero;* (2) *the algebraic sum of the products of the forces into their respective distances, from some fixed point, measured along a line drawn through that point across their directions, equals zero.*

These propositions are most important in the solution of problems. It is often found convenient to take the point O in the line of one of the forces, in which case the distance of that particular force from O and the corresponding term $P\,x$ in equation ii. both vanish.

§ 182. **Examples.**—(1) A beam $A\,B$, 10 feet long, the weight of which is 10 lbs., and acts at its middle point,[1] is supported on two props, 1 foot and 2 feet from the ends A and B respectively. From the extremity A a mass of 20 lbs. hangs; find the pressure in pounds-weight on each prop.

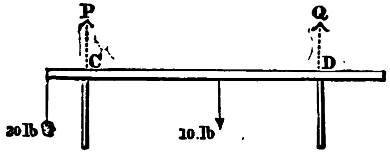

FIG. 83

Here we have two forces acting vertically downwards, viz. 10 lbs.-weight and 20 lbs.-weight, and the reactions of the two props, which are together equivalent to these forces, acting vertically upwards.

Let P and Q be the reactions of the two props, which are equal in magnitude and opposite in direction to the pressures on them.

Then $P + Q = 30$. Let the fixed point from which the distances are to be measured be at D. Then we have

[1] In all statical questions connected with beams and heavy bodies the 'weight of the body' is regarded as acting at some definite point in it. The reasons for this will be explained in the next chapter.

$$P \times 7 + Q \times 0 - 20 \times 8 - 10 \times 3 = 0 \quad . \; . \; . \; . \quad (1)$$

$$\therefore 7P = 190,$$

$$\therefore P = 27\tfrac{1}{7} \text{ and } Q = 2\tfrac{6}{7}.$$

It should be noticed that by measuring distances from D the unknown force Q drops out of equation (1); hence the convenience of choosing D.

(2) Bodies weighing 3 oz., 4 oz., and 5 oz. are hung on a rod $A\,B$, 12 inches long, at distances of 1 inch, 2 inches, and 8 inches from the end A (fig. 84). If the weight of the rod be 6 oz. and act at its centre, find

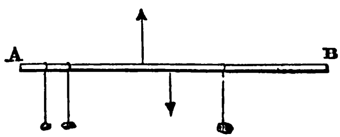

FIG. 84

where the rod must be suspended that it may rest horizontal.

Here we have a force of 3 oz. wt. at a distance of 1 inch from A,

a force of 4 oz. wt. at a distance of 2 inches from A

„ „ 5 „ „ „ 8 „ „

and „ „ 6 „ „ „ 6 „ „

The only force acting in the opposite direction is the pull of the string, which is equal to $3 + 4 + 5 + 6 = 18$ oz. wt., and acts at a distance x to be found from the end A;

$$\therefore 18x = 3 + 4 \times 2 + 5 \times 8 + 6 \times 6 = 87$$

$$\therefore x = \frac{87}{18} = 4\tfrac{5}{6} \text{ inches from } A,$$

(3) Four like parallel forces, the magnitudes of which, are 2, 3 4, and 5, act at the four corners A, B, C, D (fig. 85) of a square; find the position of their resultant.

It is sometimes convenient to combine first one pair of forces, and then another pair, and finally to combine the two resultants. The proper choice of these combinations will often materially simplify the calculation in a particular problem.

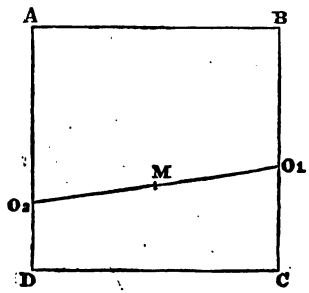

FIG. 85

The resultant of the forces 3 and 4 at B and C is a force of 7 acting at O_1, where $B\,O_1$ equals $\frac{4}{7}$ $B\,C$. The resultant of 2 and 5 is a force of 7 acting at O_2, where $A\,O_2$ is $\frac{5}{7}$ of $A\,D$. Join $O_1\,O_2$ and the final resultant will be a force of 14 acting at the point M which bisects $O_1\;O_2$. Hence the centre of the forces is situated at a point equally distant from $A\,D$ and $B\,C$ and $\frac{9}{14}$ of one of the sides from $A\,B$ and $\frac{5}{14}$ from $D\,C$.

§ 183. **Resolution of a force into Parallel Components.**—A force F may be resolved into two parallel components P and Q, provided that $P \times P$'s distance from $F = Q \times Q$'s distance from

F; and either or both of these components may be again resolved, provided that the relation between the distances of the components from the original force is preserved.

§ 184. **Example.**—A mass of weight W rests on a triangular table at a point in the line joining one of the angular points with the middle of the opposite side, and at a distance from the angular point equal to twice its distance from the side. The table is supported on three props at its angular points; find the pressure on each prop.

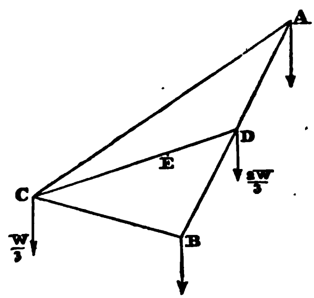

FIG. 86

Let the force W at E be resolved into two parallel forces at D and C. Then, since $CE = 2\,ED$, the component at D equals twice the component at C, and $\frac{2\,W}{3}$ acts at D and $\frac{W}{3}$ acts at C. But $\frac{2\,W}{3}$ at D is equivalent to $\frac{W}{3}$ at A and $\frac{W}{3}$ at B; therefore the force W exerts a pressure of $\frac{W}{3}$ on each prop.

§ 185. **Couples.**—In § 176, II., we found the resultant of two *unlike* parallel forces P and Q, on the supposition that Q was *greater* than P. Let us see what would have happened if Q had been *equal* to P. In this case $A\,s$ and $B\,s'$ (fig. 78) would have been parallel, and the point O through which the resultant passes would have been infinitely distant, and the proof on p. 266, which depends on the meeting of $B\,s'$ and $s\,A$, would not hold. It thus appears that two equal and opposite parallel forces have no resultant, i.e. there is no single force tending to produce translation that can replace them. Such a pair of forces is called a *couple*, and the perpendicular distance between the forces is called their arm. The effect of a couple is to produce *rotation*, and it generally happens that one of the forces is replaced by the reaction of a fixed point about which the body is free to rotate. The conditions of equilibrium of a body free to rotate about a fixed point will be considered in the two following Lessons.

EXERCISES XXI

1. Parallel forces are applied at two points 5 ins. apart, and are kept in equilibrium by a third force 3 ins. from the one and 2 ins. from the other. What is the ratio of the forces?
2. Two men, of the same height, carry on their shoulders a pole 6 feet long, and a body weighing 121 lbs. is slung

on it, 30 ins. from one of the men; what portion of the weight does each man support?

3. A beam the weight of which is 10 lbs. acting at its middle point is supported on two props at the end of the beam. If the length of the beam be 5 feet, find where a mass of 30 lbs. wt. must be placed, so that the pressing forces on the two props may be 15 lbs. wt. and 25 lbs. wt. respectively.
4. Two weights of 3 oz. and 5 oz. act at the ends of a rod $A\,B$, 12 ins. long, and a third weight of 6 oz. acts at a point 3 ins. from A; find the position of the resultant.
5. Equal weights are at the corners of a triangle; find the point at which it must be suspended to rest horizontally.
6. Equal forces act along $A\,B$, $B\,C$, $D\,C$, $A\,D$, the sides of a square; find the resultant.
7. The ratio of two unlike parallel forces is $\frac{4}{5}$, and the distance between them is 10 centimetres; find the position of the resultant.
8. The resultant of two unlike forces is a force of 6 poundals, and acts 8 ins. from the greater force, which is 10 poundals; find the distance between them.
9. Forces of 3, 4, 5, and 6 units act at distances of 3 ins., 4 ins., 5 ins., 6 ins. from the end of a rod; at what distance from the same end does the resultant act?
10. A circular table rests on one central leg, and a body weighing 10 kilograms is placed midway between the centre and circumference; where must a weight of 8 kilograms be applied so as to preserve equilibrium?
11. A plank weighing 10 lbs. rests on a single prop at its middle point; if it be replaced by two others on each side of it, 3 feet and 5 feet from the middle point, find the force pressing on each.
12. Bodies weighing 4 lbs. and 12 lbs. are fixed to the two ends respectively of a weightless rigid rod 30 ins. long, and at 10 ins. from each end are bodies weighing 6 lbs. and 8 lbs. respectively, where must the rod be suspended to rest horizontally?

13. A weightless rod is suspended at a point 3 ins. from one end, and a body weighing 10 lbs. is hung from the same end; if the rod be 15 ins. long, find the weight that must be applied at the other end to maintain equilibrium.
14. If the weight of the rod be 2 lbs. acting vertically downwards at its middle point, what weight must then be applied at the further end?
15. Four vertical forces 4, 6, 7, 9 act at the four corners of a square; find where the resultant acts.
16. Find the centre of like parallel forces, 3, 2, 5, 7, which act at equal distances apart along a straight rod 12 ins. long.
17. A horizontal straight bar 6 feet long and weighing 16 lbs. is supported at each end, and a mass weighing 48 lbs. is hung at 2 feet from one end; find the force pressing upon each of the supports.
18. Three parallel forces of 4, 5, and 7 units are applied respectively at the centre and two extremities of a rigid bar; find their resultant in magnitude and position.

XXV. *Forces producing Rotation—Moments*

§ 186. We have hitherto considered the tendency of forces to produce *translation*, or motion from one place to another in the bodies on which they act; we have now to treat of the tendency of forces to produce *rotation* about some fixed point or axis.

If a point in a body be supposed fixed, so that the body cannot move out of its own place, but is

free to rotate about that point, a force applied at any other point and in a direction that does not pass through the fixed point will produce *rotation.*

§ 187. The rotatory effect of a force depends on :—

First. The magnitude of the force.

Secondly. The perpendicular distance of its line of action from the *fixed* point.

In closing a door a small force applied at the handle will produce the same effect as a much larger force applied at a point nearer to the hinge.

§ 188. **Moment of a Force** —The tendency of a force to produce rotation about a fixed point is

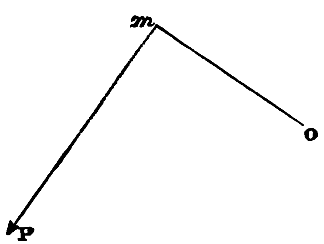

FIG. 87

called its *moment* about that point, and is measured by the product of the number of units of force into the number of units of length in the perpendicular drawn from the fixed point on to the line of action or direction of the force. Thus if P be

a force tending to cause a body to rotate about tho point O, and $O\,m$ be the perpendicular drawn from O on to its direction, then $P \times O\,m$ (i.e. the number of units of force in P multiplied by the number of units of length in $O\,m$) measures the rotatory tendency of the force, and is called the moment of the force P about the point O.

§ 189. **Geometrical Measure of the Moment of a Force.**—If $A\,B$ represent the magnitude of

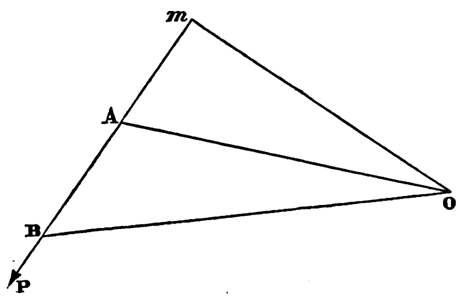

FIG. 88

P, then the measure of the moment is $A\,B \times O\,m$, which is equal to twice the area of the triangle $A\;O\;B$. Hence the moment of a force about a point may be measured by twice the area of the triangle which has the straight line representing the force for a base, and the fixed point for an apex.

If two triangles having the same apex can be

shown to be equal, the moments of the forces, represented by the bases, about that apex will be equal also.

§ 190. The moment of a force about a point in its own line of action is evidently nothing, since no perpendicular can be let fall on a line from a point in the same line.

§ 191. **Positive and Negative Moments.**—In considering the action of forces we found it convenient to distinguish between positive and negative forces, according as they tended to move the body to the right or to the left of a fixed point. In the same way it is found desirable to speak of positive and negative moments. The moment of a force may be said to be positive when it tends to produce rotation in the direction in which the hands of a clock move, and negative when its tendency is in the reverse direction.

§ 192. **The sum of the moments of two or more forces about any point is equal to the moment of the resultant of the forces about that point.**

Let $A\,B$, $A\,C$ represent two forces P and Q, and $A\,D$ their resultant R.

Take any point O in the plane of P and Q, and join $A\,O$, $O\,B$, $O\,C$, and $O\,D$, and from B, C, D

draw to $A\,O$ perpendiculars $B\,b$, $C\,c$, $D\,d$. Then the moments of P, Q, and R about O are measured by the areas of the triangles, $A\,O\,B$, $A\,O\,C$, and $A\,O\,D$ respectively. But since the base $A\,O$ is common to these three triangles, and the perpen-

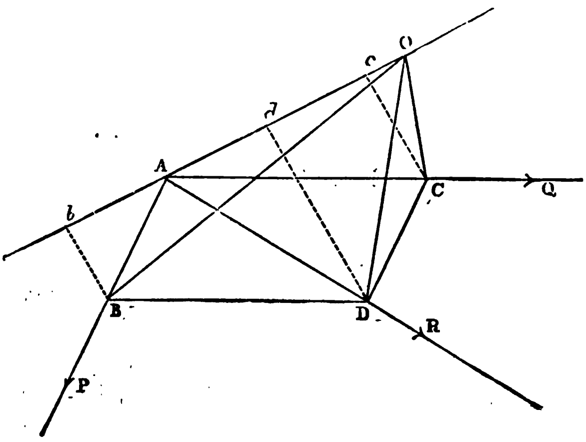

FIG. 89

dicular $D\,d$ is equal to $B\,b + C\,c$, it follows that the area of the triangle $A\,O\,D$ is equal to

$$\triangle\, A\,O\,B + \triangle\, A\,O\,C\,;$$

$\therefore$ the moment of R about O is equal to the sum of the moments of P and Q about O.

If we have a third force S acting at A, it follows that the moment of the resultant of R

and S about O is equal to the sum of the moments of R and S about O, and so for other forces.

Hence, if P_1, P_2, P_3 . . . be co-planar and concurrent forces, and $P_1 p_1$, $P_2 p_2$, $P_3 p_3$ . . . represent their several moments about a point in the plane of the forces, and Rr the moment of the resultant of all the forces about the same point, then

$$Rr = P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots$$

§ 193. **If any point be taken in the line of action of the resulant of two Forces the Moments of the Forces about this point will be equal.**—This proposition follows at once from the preceding, for since the moment of the resultant about a point in its own line of action is zero, we have $P_1 p_1 + P_2 p_2 = 0$, or $P_1 p_1 = - P_2 p_2$, i.e. the moments are equal in magnitude but opposite in direction.

The proposition may, however, be proved independently, thus :—Let P and Q be two forces acting at C, and let R be their resultant. In the line of action of R take any point O, and draw OA, OB parallel to the directions of Q and P respectively. Then CO, CA, and CB will represent R, P, and Q, wherever the point O may be taken. But the moment of the force CA about the point O is represented by twice the area of the

triangle CAO (§ 189), and the moment of CB by twice the triangle CBO, and

$$\text{the triangle } CAO = \text{the triangle } CBO;$$

$\therefore$ the moment of P about O equals the moment of Q about the same point.

$$\text{Or,} \qquad P \times Om = Q \times On.$$

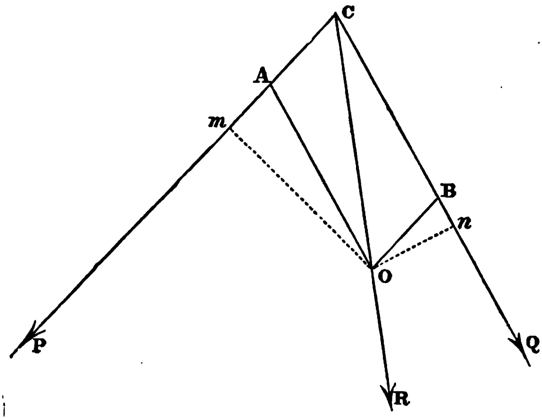

FIG. 90

Conversely, it is easily seen, that if the moments of the two forces about any point are equal, i.e. if the triangle CAO is equal to the triangle CBO, the point O must lie on the line of action of the resultant of the forces.

§ 194. **Equilibrium of two Moments.**—If two forces act at different points of a body which is

free to rotate about a fixed point they will produce equilibrium when their moments about that point are equal and opposite. Now, we have seen that the moments of two forces are equal and opposite about any point in the line of action of their resultant. Hence a body acted upon by two forces will be in equilibrium when the fixed point in the body is in the line of action of the resultant.

§ 195. **Application to Lever.**—The lever has been defined as a rigid rod capable of turning

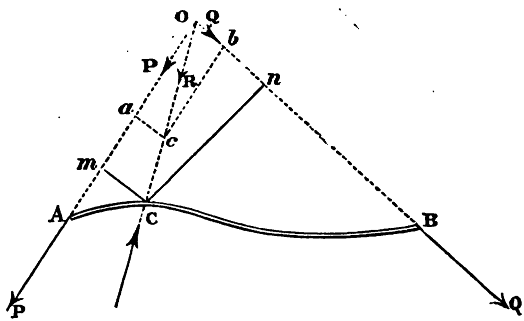

FIG. 91

about a fixed point called the Fulcrum. We are now able to find the conditions of equilibrium on the lever, and the relation between the forces acting upon it in other cases than those already considered. For if P and Q be two forces acting at the points A and B of a rigid rod of any form,

and if the rotatory tendencies or moments of these forces are equal about the point C or fulcrum, i.e. if they produce equilibrium, then the resultant of P and Q passes through C. If, therefore, the directions of P and Q be produced to O (fig. 91), and $O a$ and $O b$ represent the two forces, and $O c$ their resultant, and if $O c$ be produced, then the point C or the position of the fulcrum will be in $O c$ produced, and the force pressing on the fulcrum will be represented by $O c$. And if $C m$ and $C n$ be the perpendiculars, drawn from C to the directions of P and Q, $P \times C m = Q \times C n$. Hence the following propositions hold good with respect to the lever, when there is equilibrium, whether the lever be straight or bent, and whether the forces are perpendicular to the arms or not.

(1) The resultant of the two forces passes through the fulcrum of the lever, and is equal to the force pressing on the fulcrum.

(2) The algebraic sum of the moments of the forces about the fulcrum equals zero.

This latter proposition is generally known as the *principle* of the lever.

§ 196. **Examples.**—(1) A bent lever without weight consists of two arms, one of which is twice as long as the other, inclined to each other at an angle of 120°; find the ratio of the weights of the bodies that must be suspended from their ends, so that the lever may rest with the shorter arm horizontal.

Let P and Q be the two weights. Taking moments about C, we have

$$P \times AC = Q \times Cn;$$

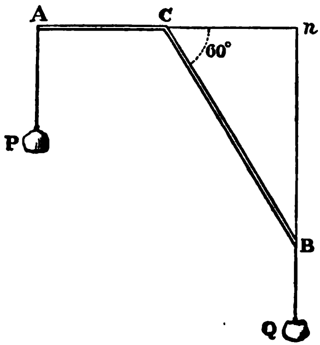

FIG. 92

and since the angle BCn is 60°

$$Cn = \tfrac{1}{2} CB = AC;$$

$$\therefore P \times AC = Q \times AC, \text{ or } P = Q.$$

(2) A rod the weight of which is 10 lbs. and acts at its middle point moves at one end about a hinge, and is supported at the other end by a cord attached to a point, vertically over the hinge, and at a distance from it equal to the length of the rod; find the *pull* of the cord when the rod rests in a horizontal position. Let AB be the rod movable about a hinge at A. Let CB be the cord, then $AC = AB$. Let W be the weight of the rod at G, and let T be the pull in BC.

Then in equilibrium the moment of W about A must be equal to the moment of T about A.

Or if $A\,m$ be drawn perpendicular to $C\,B$

$$T \times A\,m = W \times A\,G,$$

and $$A\,m = \frac{A\,B}{\sqrt{2}}, \text{ and } A\,G = \frac{A\,B}{2}$$

$$\therefore \frac{T}{\sqrt{2}} = \frac{W}{2},$$

$$\text{or } T = 5\sqrt{2} \text{ lbs.}$$

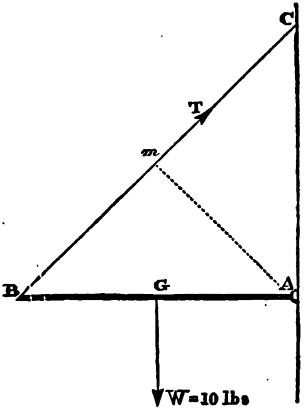

FIG. 93

§ 197. **Balances.**—When a lever is employed to determine the weight of a substance it is called a balance. This may consist of a beam supported at its middle point, with pans hanging from either end, one of which holds the substance to be weighed and the other the weights; or the arms of the beam may be of different but fixed

lengths, and the weight may slide upon it ; or the arms themselves may vary by the movement of the fulcrum.

§ 198. **The Common Balance.**—Here the arms are equal, and the beam is nicely balanced on knife-edges.

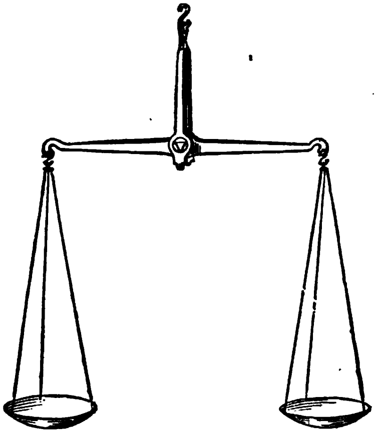

FIG. 94

A balance should be so constructed that—

1. When bodies of equal weight are placed in the scale-pans, the beam should be perfectly horizontal. The balance should be *true*.

2. When the weights of the bodies differ but slightly, the *deviation* of the beam from the hori-

zontal position should be considerable. The balance should be *sensitive.*

3. When disturbed the balance should quickly resume its original position. The balance should be *stable.*

To test if a balance be *true*, place a certain mass in one scale-pan and some substance that balances it in the other. Then interchange the contents of the two pans, and if the beam remain horizontal the balance is true.

§ 199. The *true* weight of a body can be ascertained with a *false* balance, if the weights used are correct, in the following manner :—

Let W be the *real* weight of a body, and suppose it is found to weigh a lbs. when placed in one scale-pan, and b lbs. when placed in the other. Let x and y be the unknown lengths of the arms of the balance. Then, by the principle of moments,

$$W \times x = a \times y \text{ and } W \times y = b \times x,$$

$$\therefore\ W^2 \times xy = ab \times xy \ \therefore\ W = \sqrt{ab}.$$

Or, the *true* weight is the square-root of the product of the *apparent* weights when the body is weighed in each scale-pan.

§ 200. **The Common, or Roman Steelyard.**— This is a balance with unequal arms. The body to be weighed hangs from the end of one arm of the

beam, and the weight employed to measure it slides on the other arm of the beam.

Let the beam be suspended at C (fig. 95), and let w be its weight acting at G (see note, p. 273), so that the weight P at the point O is able to keep the beam in equilibrium. Hence $w \times CG = P \times CO$.

Suppose a body weighing W suspended from

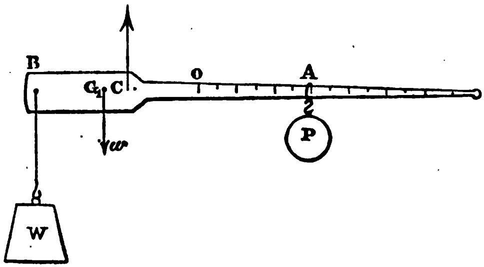

FIG. 95

B, and that the beam is in equilibrium when the movable weight P is at A. Then

$$W \times BC + w \times GC = P \times AC$$
$$= P \times AO + P \times OC.$$
$$\text{But } w \times GC = P \times OC;$$
$$\therefore W \times BC = P \times AO,$$

or
$$\frac{W}{P} = \frac{AO}{BC}.$$

Hence, the distance from the point O at which the

weight P must hang varies directly with the magnitude of W.

If $OA = 1$ inch when $W = 1$ lb.
$OA = 2$ inches „ $W = 2$ lbs.
$OA = 3$ „ „ $W = 3$ lbs., and so on.

§ 201. **The Danish Steelyard.**—This instrument consists of a straight bar with a heavy knob at one end (fig. 96). The body to be weighed is suspended at the other end, and the fulcrum O is moved until the moment of the weight of the body is equal to the moment of the weight of the beam.

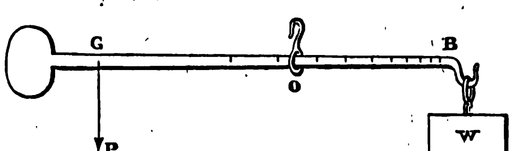

FIG. 96

Let P equal the weight of the beam at G, and W the weight of the body suspended at B; then

$$P \times GO = W \times OB,$$

$$\therefore P \times (GB - OB) = W \times OB,$$

$$\therefore OB = \frac{P}{P + W} \cdot GB.$$

Hence, P and GB being fixed quantities, the beam is graduated by making W equal to 1 lb., 2 lbs., 3 lbs. successively.

Thus, if $W = P$, then $OB = \frac{1}{2}\,GB$;
" $W = 2P$ " $OB = \frac{1}{3}\,GB$;
" $W = 3P$ " $OB = \frac{1}{4}\,GB$, and so on.

§ 202. Other balances are frequently formed by altering the inclination of a bent lever, and by indicating the corresponding change in the moment of the weight of the lever about its fulcrum. The common letter-weight, shown in fig. 97, is an ex-

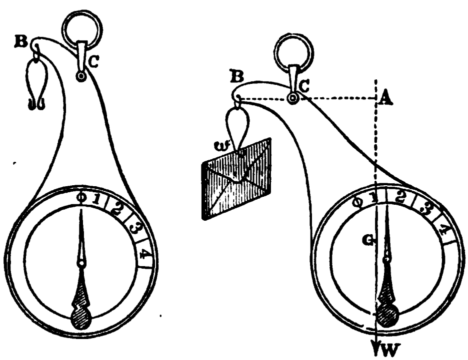

FIG. 97

ample of this kind of balance. The fulcrum is at C, near the shorter arm of the lever. The letter to be weighed is suspended from B. When there is no weight at B, the point G, at which the whole weight of the instrument acts, is immediately under C, the point of suspension, and the index, which always remains vertical, points to zero. But if anything be hung from B, the balance

assumes a different position, and the moment of its weight about C increases. If W be the weight of the instrument acting at G, and w be the weight of the body at B, we have $W \times AC = w \times BC$, when equilibrium exists. If bodies weighing $\frac{1}{2}$ oz., 1 oz., &c., be separately suspended from B, and if the corresponding positions of the vertical index be marked, these marks serve as a scale of weights.

§ 203. **General Properties of Moments.**—Since the moment of a force measures its tendency to produce rotation about a fixed point, it is evident that in order that a body, acted upon by several forces, may be in equilibrium, the various tendencies to rotation must counterbalance one another, or the sum of the positive moments with respect to any point must equal the sum of the negative moments. Hence, *when several forces act at different points of a rigid body, and are in equilibrium, the algebraic sum of the moments of the forces about any point must vanish.* This proposition follows directly from that already proved (§ 192), that the moment of the resultant is equal to the sum of the moments of the forces; for where there is equilibrium the moment of the resultant is *zero,* and therefore the sum of the moments of the forces is zero.

§ 204. Moreover, if the fixed point about

which rotation is supposed to take place be in the line of action of the resultant, the moment of the resultant about this point is zero, and consequently *the algebraic sum of the moments of a number of forces about any point in the line of action of their resultant is zero.*

This proposition has also been separately proved in the case of two forces meeting at a point (§ 193).

EXERCISES XXII

1. Two equal rods are jointed together and form a right angle. They move freely about their common point. Neglecting the weights of the rods, find the ratio of the weights of the bodies that must be suspended from their extremities that one of the rods may be inclined to the horizon at 60°.
2. A rod AB moves about a fixed point B. Its weight W acts at its middle point, and it is kept horizontal by a string AC that makes an angle of 45° with it. Find the pull of the string.
3. A rod 10 inches long can turn freely about one of its ends; a body weighing 4 lbs. is slung to a point 3 inches from this end; what vertical force at the other end is required to support it?
4. If the rod, in the preceding question, be held by a string attached to the free end of the rod and inclined to it at an angle of 120°, find the pull in the string when the rod is horizontal.
5. Two weights of 3 lbs. and 4 lbs. act at the extremities of a straight lever 12 inches long, and inclined to it at angles of 120° and 135° respectively; find the position of the fulcrum.
6. A uniform lever is bent so that its arms make an angle of 150° with each other, if the longer arm

remains horizontal when a body weighing 3 oz. hangs from its extremity, and when a body weighing 12 oz. hangs from the end of the shorter arm; find the ratio of the arms.

7. Find the true weight of a body which is found to weigh 8 oz. and 9 oz. when placed in each of the scale-pans of a false balance.

8. In a common steelyard the weight of the beam is 10 lbs., and acts at a distance of 2 inches from the fulcrum; find where a weight of 4 lbs. must be applied to balance it.

9. A Danish steelyard is 36 inches long, and its weight, 2 lbs., acts at a point 6 inches from one end. At the other end is hung a body weighing 10 lbs.; find the position of the fulcrum.

10. If in the preceding question the fulcrum were $7\frac{1}{2}$ inches from the thin end, what would be the weight of the body?

11. A beam 3 feet long, the weight of which is 10 lbs., and acts at its middle point, rests on a rail, with 4 lbs. hanging from one end and 13 lbs. from the other; find the point at which the beam is supported; and if the weights at the two ends change places, what weight must be added to the lighter to preserve equilibrium?

12. Show that the sum of the moments of two forces represented by the two sides of a triangle about any point in the base is constant.

XXVI. *Couples—General Conditions of Equilibrium of Forces in one Plane*

§ 205. **Couples.**—We have seen that when two equal dissimilar forces, not in the same straight line, act upon a body, they produce a *couple*, the

effect of which is to cause *rotation.* The measure of this couple is its moment, or the product of either force forming the couple into its arm, i.e. the perpendicular distance between them.

§ 206. **The moment of a couple is equal to the algebraic sum of the moments of the two forces forming the couple about any point in the plane of the forces.**

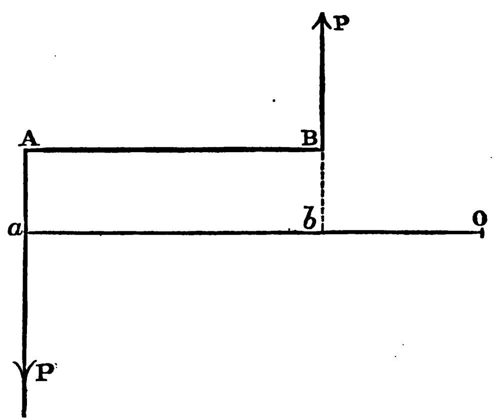

FIG. 98

Let P and P be two forces acting at A and B, where $A\,B$ is the arm of the couple.

Then $P \times A\,B$ is the moment of the couple. Let O be any point in the plane of the forces, and $O\,b\,a$ a line drawn at right angles to their direction, then

$P \times AB = P \times (aO - bO) = P \times aO - P \times bO =$ the algebraic sum of the moments about O.

§ 207. **To find the Resultant of two Couples in the same Plane.**—Since the measure of a couple is the product of either force into the arm, it is clear that one couple may be replaced by another of the same *sign*, of equal moment.

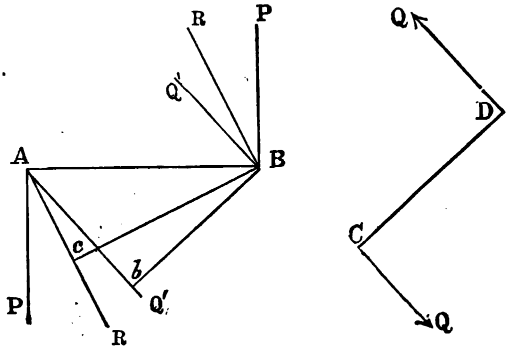

FIG. 99

Let $P \times AB$ and $Q \times CD$ (fig. 99) be the moments of two distinct couples. At A and B apply two equal forces, Q' parallel to the direction of Q, and of such magnitude that $Q' \times Bb = Q \times CD$.

Then the couple $Q' \times Bb$ is equivalent to the couple $Q \times CD$. The resultants of P and Q' at A and of P and Q' at B can be found by the parallelogram of forces. Let R be the resultant. Then

R and R acting at A and B form a couple, the moment of which is $R \times B c$, which is equivalent to the two couples $P \times A B$ and $Q \times C D$. In the same way may be found the resultant of the couple $R \times B c$ and any other couple, and hence any number of couples may be compounded into a single couple.

§ 208. **Any number of forces acting at different points of a body in the same plane can be reduced to a single force and a single couple.**

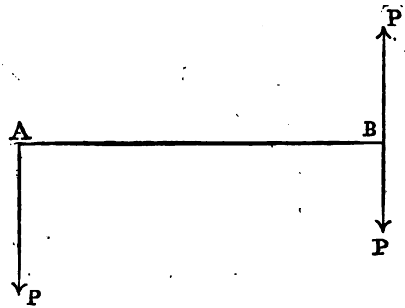

FIG. 100

Let P be any force acting at A. Across the direction of P draw $A B$ at right angles to it, and at B apply two equal and opposite forces P and P, in a direction parallel to that of P at A. These two new forces being equal and opposite, and acting at the same point, will produce no effect. But the system now consists of a couple $P \times A B$, and a force P at B. Hence a force acting at one

point can be replaced by a force acting at some other point and a couple. Similarly any other force Q can be replaced by a force Q at B and a couple, and so with any number of forces. But all such forces acting at B have a resultant which is a single force acting at B; and all the couples can be reduced to a single resultant couple. Hence any number of forces acting at different points in the same plane of a rigid body can be reduced to a single resultant force and a single resultant couple.

§ 209. **The resultant couple is equal to the algebraic sum of the component couples.**

We have seen (§ 206) that the moment of a couple is equal to the algebraic sum of the moments of the component forces about any point in their plane. Taking the resultant couple $R \times AB$, we have $R \times AB$ equal to the algebraic sum of the moments of R and R about any point in the same plane. But the moment of R about any point is equal to the algebraic sum of the moments of the component forces about that point (§ 192). Hence the moment of the resultant couple is equal to the algebraic sum of the moments of the component couples.

§ 210. **Conditions of Equilibrium of any number of forces acting in one plane at different**

points of a rigid body.—We are now in a position to give the results of the solution of the problem (with respect to forces in one plane) which we stated at the opening of this chapter to be the problem of Statics.

We have seen that when any number of forces act at different points of a rigid body and in different directions they can be reduced to a single resultant force and a single resultant couple. In order that the system of forces may be in equilibrium, i.e. may cause neither *displacement* nor *rotation*, the resultant force must be *zero* and the moment of the resultant couple must be *zero*. Let R be the resultant force and $R \times A\,B$ the moment of the resultant couple. Then for equilibrium $R = 0$ and $R \times AB = 0$. And since $R^2 = X^2 + Y^2$, where X and Y are the algebraic sums of the components of the forces in directions at right angles to each other, we must have $X = 0$ and $Y = 0$. And $R \times AB$ equals the algebraic sum of the moments of the forces about any point in the same plane. Hence, we have as the general conditions of equilibrium,

$$(1)\ X_1 + X_2 + X_3 + \ldots = 0,$$
$$(2)\ Y_1 + Y_2 + Y_3 + \ldots = 0,$$
$$(3)\ P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots = 0,$$

where $X_1, X_2 \ldots$ and $Y_1, Y_2 \ldots$ are the components of the forces $P_1, P_2 \ldots$ in any two directions at right angles to each other, and $P_1 p_1$,

P_2p_2 . . . are the moments of P_1, P_2 . . . about any point in the same plane, the usual conventions in regard to *sign* being employed in all these equations.

§ 211. **Recapitulation.**—In investigating the conditions of equilibrium when two or more forces in the same plane act simultaneously on a body we have arrived at the following results :—

In order that there may be equilibrium when—

(i) Two forces act on a body,

1. they must be equal in magnitude, 2. opposite in direction, and 3. act at the same point.

(ii) Three forces act on a body, they must

1. either pass through a point, and be capable of being represented by the sides of a triangle taken in order ;
2. or be parallel, in which case the algebraic sum of the forces and of their moments about any point must vanish.

(iii) Several forces act on a body and

1. pass through a point,

   They must be capable of being represented by the sides of a polygon taken in order.

2. are parallel,

   The algebraic sum of the forces and

their moments about any point must vanish.

3. act at different points, and in different directions,

   The algebraic sums of the components of the forces in each of two directions at right angles must separately vanish; and the algebraic sum of the moments of the forces about any point in their plane must likewise vanish.

The three conditions involved in each of these cases are sufficient for the equilibrium of a body under the action of any number of forces *in one plane.*

§ 212. In the solution of problems condition (iii) 1. very frequently enables us to obtain a geometrical representation of what is sought, although the calculation necessary to obtain a numerical solution of the problem may in many cases involve a higher knowledge of mathematics than the student is supposed to possess.

§ 213. **Examples.**—(1) A beam the weight of which acts at its middle point hangs from a nail by two strings of given length attached to its extremities; required the position of the beam in equilibrium.

Let the weight act at C, then the forces in equilibrium are the pulls in the two strings and the

weight; and *since these three forces must pass through a point*, the beam must so rest that the point C is vertically under the nail.

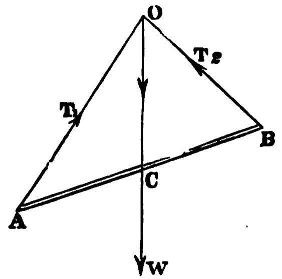

FIG. 101

(2) A rod the weight of which acts at its middle point is placed on two smooth inclined planes; required the position of the rod in equilibrium.

Here the forces acting are W at G, and R and S the reactions of the planes; and since the planes are

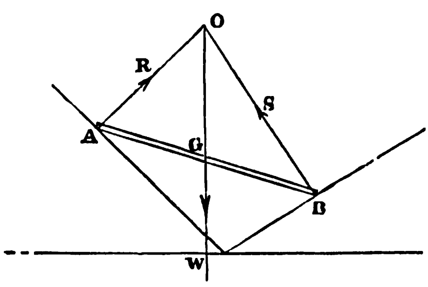

FIG. 102

supposed to be perfectly smooth, these reactions are perpendicular to the surfaces. These three forces must

pass through a point, and the problem of finding the position of equilibrium can be solved by a purely geometrical process, since the angles which R and S respectively make with the vertical are equal to the inclinations of the planes to the horizon.

(3) A ladder the weight of which is W and acts at a point one-third of its length from the foot is made to rest against a smooth vertical wall, and inclined to it at an angle of 30°, by a force applied horizontally to the foot; find the force.

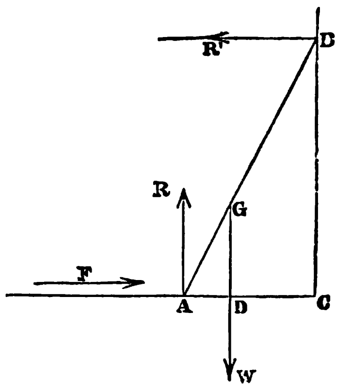

FIG. 103

Let F be the force required. Then the forces acting are W at G, the reaction of the ground R at A, and the reaction of the wall R' at B.

These form two couples which preserve equilibrium. By § 211 (iii) 3, $F = R'$ and $R = W$, and, by taking moments about A, we have

$$W \times AD = R' \times CB,$$

$$\text{and } AD = \frac{AG}{2} = \frac{AB}{6}, \text{ and } CB = \frac{AB\sqrt{3}}{2}.$$

Also $R' = F$,

$$\therefore \frac{W}{6} = F \cdot \frac{\sqrt{3}}{2}, \text{ or } F = W \frac{1}{3\sqrt{3}}.$$

(4) A flexible cord has four masses weighing 1 lb. each attached to it at points one foot apart; the ends of the cord pass over two smooth pegs more than 3 feet apart, in the same horizontal line, and masses weighing 3 lbs. each are attached to the ends. Find the position of equilibrium and the pulls in each portion of the cord.

This problem is typical of a class of problems referred to in §§ 155 and 212, which may be readily solved by graphical methods, and the solution of which could only be obtained with much greater difficulty by numerical calculation.

Let E and F (fig. 104) be the pegs, and A, B, C, and D the points on the cord to which the 1 lb. masses are attached. The distances $A\,B$, $B\,C$, and $C\,D$ are

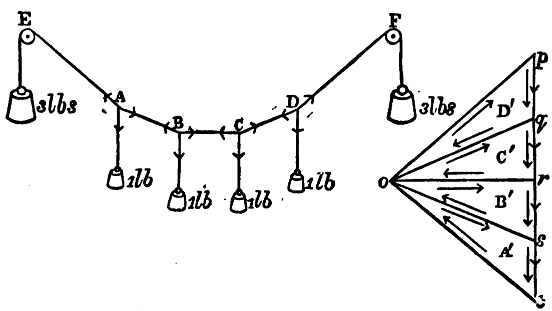

Fig. 104

equal. From symmetry, we see that $B\,C$ must be horizontal, but its position and the positions of the other parts of the cord have yet to be found.

To obtain the magnitudes and directions of the pulls, where still unknown, on the various parts $A\,E$, $A\,B$, $B\,C$, &c., of the cord, we must construct a force-polygon (in this case a triangle) for each of the points A, B, C, and D. Draw a vertical line $p\,q\,r\,s\,t$, and mark off equal distances $p\,q$, $q\,r$, &c., to represent the vertical 1 lb. weight forces. Then if $q\,r$ represents the vertical force at C, $r\,o$ will represent the *direction* of the pull in $C\,B$ at C. To find the magnitude of this pull we must find the position of o, and then $o\,q$ will represent the direction and magnitude of the pull at C in $C\,D$; but this pull is equal and opposite to the pull in $D\,C$ at D, and $p\,q$ represents the magnitude and direction of the vertical force at D, therefore $o\,p$ (which closes the polygon $p\,q\,o$) must represent the magnitude and direction of the pull in $D\,F$ at D. This pull we know to be 3 lbs. weight, and therefore the line $p\,o$ is taken 3 times as long as $p\,q$, and is placed so that its end o rests on the horizontal line $r\,o$ already drawn. Thus the point o is found, and joining $o\,q$, $o\,s$, and $o\,t$, we have the directions and magnitudes of the forces in the various parts of the cord. The magnitudes can be measured off to scale, and the position of equilibrium of the cord found by drawing the different parts parallel to $o\,p$, $o\,q$, &c.

To assist the student in following the reasoning, the directions of the forces at A, B, C, and D are shown by arrows, and in the *Force Diagram* on the right the directions of the forces represented by each triangle are shown by arrows drawn *inside* the triangle.

EXERCISES XXIII

1. Show that if three forces act at a point, and produce equilibrium, they must lie in the same plane.
2. A rod AB, 3 feet in length, the weight of which acts at a point 1 foot from A, is hung over a smooth peg, by a cord 5 feet in length; find the length of the cord on each side of the peg when the rod is in equilibrium.
3. Two forces of 10 and 12 units act along the sides AC, CB of an equilateral triangle; find their resultant.
4. Six forces of 1, 2, 3, 4, 5, and 6 units, acting at a point, make equal angles with one another; find their resultant.
5. Two forces of 4 lbs. and 8 lbs. weight act at the end of a bar 18 inches long and make angles of 120° and 90° with it; find the point in the bar at which the resultant acts.
6. Two equal forces of 8 poundals act at an angle of 120°; a third force of 6 poundals acts at the same point, and at right angles to their plane; find the resultant of the three forces.
7. To each end of a uniform straight rod 100 inches long, and weighing 12 lbs., is fastened one end of a flexible string 140 inches long, to which a body weighing 9 lbs. is attached at a point 60 inches from one end. In what position will the rod remain in equilibrium about a pivot through the middle? and where must the pivot be placed in order that the rod may be balanced when horizontal?
8. The extremities of the horizontal diameter of a circular disc, weighing 6 oz., are nailed against a wall, and to a point in the edge of the disc at $\frac{1}{12}$ of the whole circumference from one of the nails a weight of 4 oz. is applied; find the force pressing upon each nail.

9. A rod AB, 5 feet long, without weight, is hung from a point C by two strings which are attached to its ends and to the point; the string AC is 3 feet and BC is 4 feet in length, and a body weighing 2 lbs. is hung from A, and a body weighing 3 lbs. from B; find the pulls of the strings and the condition of equilibrium.

10. A body weighing 24 lbs. is suspended by two flexible strings, one of which is horizontal, and the other is inclined at an angle of 30° to the vertical direction. What is the pull in each string?

11. The weight of a window-sash 3 feet wide is 5 lbs., each of the weights acting on the cords is 2 lbs.; if one of the cords be broken, find at what distance from the middle of the sash the hand must be placed to raise it with the least effort.

12. A light smooth stick 3 feet long is loaded at one end with 8 oz. of lead; the other end rests against a smooth vertical wall, and across a nail which is 1 foot from the wall. Find the position of equilibrium and the force pressing on the nail and on the wall.

EXAMINATION QUESTIONS VII

1. A pole 12 feet long weighing 25 lbs. rests with one end against the foot of a wall, and from a point 2 feet from the other end a cord runs horizontally to a point in the wall 8 feet from the ground; find the 'tension' of the cord and the pressure of the lower end of the pole.—*Univ. of Lon. Matric.*, Jan. 1871.

2. A solid roller, with an axle projecting from one end, is suspended horizontally by two vertical cords—one of them attached to the end of the roller opposite to the axle, the other to the middle of the axle; the roller is 4 feet long, and weighs 27 lbs.; the axle is 1 foot long, and weighs 1 lb. Find the weight supported by each cord.—*Ib.* June 1871.

3. A man wheels a loaded wheelbarrow along a level road; point out the conditions which determine how much of the total weight of load and barrow is supported by the wheel, and how much is supported by the man.—*Ib.* Jan. 1872.

4. Show how it is possible for a sailing vessel to make way in a direction different from that of the wind. Why cannot a round tub be steered at as great an angle to the direction of the wind as a long-boat?—*Ib.* June 1872.

5. When a horse is employed to tow a barge along a canal the tow-rope is usually of considerable length; give a definite reason for using a long rope instead of a short one. Show whether the same considerations hold good in relation to the length of the rope when a steam-tug is used instead of a horse.—*Ib.* June 1872.

6. If a man wants to help a cart up-hill is there any mechanical reason why he should put his shoulder to the wheel, instead of pushing at the body of the cart? And if so, show at what part of the wheel force can be applied with the greatest effect.—*Ib.* Jan. 1873.

7. Two men are carrying a block of iron, weighing 176 lbs., suspended from a uniform pole 14 feet long; each man's shoulder is 1 foot 6 ins. from his end of the pole. At what point of the pole must the heavy weight be suspended, in order that one of the men may bear $\frac{4}{5}$ of the weight borne by the other?—*Ib.* June 1873.

8. A substance is weighed from both arms of an unequal balance, and its apparent weights are 9 lbs. and 4 lbs.; find the ratio between the arms.—*Ib.* Jan. 1874.

9. If in the third system of pulleys (fig. 46) the pulleys are $1\frac{1}{4}$ in. in diameter, and the strings are parallel and attached at given points D, E, and F to the rod supporting the weight, to what point of the rod should the weight W be attached, so that the hori-

zontal direction of the rod may be maintained?—*Ib.* Jan. 1874.

10. Two uniform heavy rods, $A\,C$, $B\,C$ (fig. 105), rigidly connected together, are capable of turning round a horizontal axis at C; find the mechanical conditions which determine the position of equilibrium.—*Ib.* June 1874.

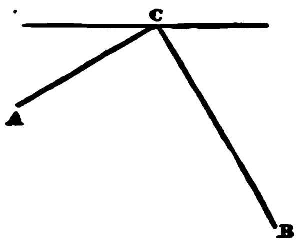

FIG. 105

11. A heavy uniform beam $A\,B$ is supported at the point C by the vertical prop' $C\,D$, its extremity A pressing against a wall $E\,F$. Determine the conditions of equilibrium, taking no account of friction.—*Ib.* Jan. 1876.

12. Three cords are tied together at a point. One of these is pulled in a northerly direction with a force of 6 lbs., and another in an easterly direction with a force of 8 lbs. With what force must the third cord be pulled in order to keep the whole at rest?—*Ib.* June 1876.

13. A sphere of wood loaded at one point with lead rests upon a plane inclined at 30° to the horizon, being prevented from sliding down by the friction of the plane. State and explain by a diagram the conditions of equilibrium.—*Ib.* Jan. 1877.

14. A body whose mass is five kilograms, resting upon a smooth plane inclined at 30° to the horizon, is

acted on by four forces: (1) its weight; (2) the reaction of the plane; (3) a force equal to the weight of two kilograms acting parallel to the plane and upwards; and (4) a force P acting at an angle of 30° to the plane. Determine the value of P that the body may be in equilibrium.—*Ib.* June 1877.

15. A ladder rests against the side of a house, and is inclined at 60° to the ground. The pressure of the ladder against the wall being equal to a force of 60 lbs., and the friction, at the same place, equal to a force of 40 lbs., find the *pressure* and *friction* at the point where the ladder rests on the ground. Find also the *weight* of the ladder.—*Ib.* Jan. 1878.

16. A uniform bar, 2 feet long, and weighing 3 lbs., is used as a steelyard, being supported at a point 4 inches from one end. Find the greatest weight which can be weighed with a movable weight of 2 lbs., and find the point from which the graduations should be measured.—*Ib.* Jan. 1879.

17. In a common steelyard whose mass is 20 lbs. the point of suspension of the body to be weighed is 4 inches from the fulcrum, while the centre of gravity of the beam is one inch from the fulcrum measured in the opposite direction. If the mass of the sliding weight be 7 lbs. find the distances from the fulcrum of the graduations marked respectively 14 lbs., 28 lbs., 56 lbs., and 112 lbs.—*Ib.* July 1879.

18. Three forces act along three of the sides of a parallelogram $ABDC$, one from A to B one from A to C, and the third from B to D; each force being proportional to the side along which it acts; the parallelogram is such that the diagonal AD is perpendicular to the side BD. Find the line of action of the resultant force, and show that its magnitude is equal to one of the given forces.—*Ib.* Jan. 1881.

19. The arms of a bent lever are at right angles to one another, and their lengths are in the ratio of 5 to 1.

The longer arm is inclined 45° to the horizon, and carries at its extremity a weight of 10 lbs. The end of the shorter arm presses against a smooth horizontal plane. Draw a figure showing the forces in action, and find the pressure between the shorter arm and the plane.—*Ib.* June 1883.

20. A uniform sphere rests on a smooth inclined plane, and is supported by a horizontal string. To what point on the surface of the sphere must the string be attached? Draw a figure showing the forces in action.—*Ib.* Jan. 1884.

21. A man carries a bundle at the end of a stick over his shoulder. If the distance between his hand and the bundle be kept constant, and the distance between his hand and shoulder be changed, how does the force on his shoulder change?—*Ib.* Jan. 1885.

22. A picture weighing 40 lbs. is hung, with its upper and lower edges horizontal, by a cord fastened to its two upper corners and passing over a nail so that the parts of the cord at the two sides of the nail make an angle of 60° with each other. Find the pull in the cord in pounds weight.—*Ib.* June 1885.

23. A piece of lead placed in one pan *A* of a balance is counterpoised by 100 grams in the other pan *B*. When the same piece of lead is placed in the pan *B* it requires 104 grams in the pan *A* to balance it. Show what is the ratio of the length of the arms of the balance.—*Ib.* June 1885.

24. An iron ball weighing 1 lb. is held up by a string, the upper end of which is fixed. If the ball be drawn aside until the string makes an angle of 30° with the vertical, and then let go, state clearly what the forces are which act upon the ball at the moment of its release.—*Ib.* Jan. 1887.

25. A picture is hung by a string fastened at two corners of its frame, and passing over a smooth nail. What are the forces producing equilibrium, and how will

the pull on the string vary as it is made longer or shorter?—*Ib.* June 1887.

26. The sides *B C*, *C A*, *A B* of a triangle are bisected at *D*, *E*, and *F* respectively; find the resultant of the forces represented by *D A*, *E B*, and *F C*.—*Ib.* June 1887.

27. A circular table, whose mass is 10 lbs., is provided with vertical legs attached to 3 points in the circumference equidistant from one another. Find the least mass which, when hung from any point in the edge of the table, will just cause it to overturn.—*Ib.* June 1888.

28. A rod *A B* is hinged at *A*, and supported in a horizontal position by a string *B C*, making an angle of 45° with the rod. The rod has a weight of 10 lbs. suspended from *B*. Find the 'tension' in the string and the force at the hinge. The weight of the rod may be neglected.—*Ib.* Jan. 1889.

29. A boat is being towed by a rope making an angle of 30° with the boat's length: the resultant pressure of the water on the boat and rudder is inclined at 60° to the length of the boat and the tension in the rope is equal to the weight of half a ton. Find the resultant force in the direction of the boat's length. —*Ib.* June 1889.

30. A uniform beam, 24 feet long and weighing 200 lbs., is supported on two props, one 6 feet from one end, the other 9 feet from the other end of the beam; calculate the pressure on each prop, when a man weighing 180 lbs. stands as near the latter end as he can without upsetting the beam.—*Ib.* Jan. 1890.

31. A weightless rod, 3 feet long, is supported horizontally, one end being hinged to a vertical wall, and the other attached by a string to a point 4 feet above the hinge; a weight of 180 lbs. is hung from the end supported by the string. Calculate the 'tension' in the string and the pressure along the rod. —*Ib.* June 1890.

32. Explain why in a common scale-pan or letter balance it does not matter whereabouts on the pan the weights are placed, although they may be sometimes near and sometimes farther off the fulcrum. *Ib.* Jan. 1891.

33. A uniform beam, 14 feet long, and weighing 120 lbs., is attached to two props, one of which is 3 feet and the other 5 feet from its centre; calculate the forces on the props when a weight of 100 lbs. is placed first at one end and then at the other end of the beam.—*Ib.* June 1891.

34. A weight of 20 lbs., suspended by a string from a peg P, is pulled aside by another string knotted to the first at a point K, and pulled horizontally. Find the force necessary to pull it until PK is 60° from the vertical; and find, at the same time, the force on the peg.—*Ib.* Jan. 1892.

35. Show that a couple has no particular point of application, but may be shifted anywhere in the same plane without disturbing the equilibrium of a body to which it is applied. Is this true of a force? Explain the difference, if any.—*Ib.* Jan. 1892.

36. ABC is a triangle whose sides, BC, CA, AB, are 7, 4, 5 units long; at A two forces act, one of 8 units from A to C, and one of 10 units from A to B; draw a straight line through A, to represent their resultant in all respects, and state the number of units of force in the resultant.—*S. & A. Dep.* 1891.

CHAPTER VIII

CENTRE OF GRAVITY—MASS-CENTRE

XXVII. *Definitions—Equilibrium of a Body on a Hard Surface*

§ 214. **Centre of Gravity.**—The weight of a body is due to the force of gravitation acting between the body and the earth. If we consider a body to be made up of a large number of small particles the force of gravitation will act between each of these particles and the earth, and will be proportional to the mass of each particle. These forces act towards the earth's centre, but having regard to the distance of the body from the earth's centre, the forces may be considered parallel. The position of the resultant of these several forces can be found by the method explained in § 180, and the point at which this resultant acts is called the centre of gravity of the body.

The centre of gravity may be defined, therefore, as the point through which the resultant of all the gravitating forces acting between the particles of the body and the earth always passes.

The position of the centre of a system of parallel forces is independent of the direction of the forces, and consequently the position of the centre of gravity would remain unchanged even it the direction of gravity were altered. It is consequently independent of the position of the body, and depends only on the distribution of the mass throughout the body.

§ 215. **Mass-Centre.**—If, instead of considering the weight of the body, we consider its mass only, the point about which the mass is evenly distributed is called the *mass-centre.* This point coincides with the centre of the system of parallel forces proportional to the masses of the several particles of which the body or group of bodies consists. As the position of this point is independent of the *magnitude* as well as of the *direction* of the forces, provided they are propor tional to the masses of the particles and all act in the same direction or are parallel, the consideration of the forces themselves may be altogether neglected; and the *mass-centre* may be defined in reference only to the relative masses of the particles and their positions, and may be said to occupy the *mean* position of the group of particles.

The centre of gravity is, therefore, a particular case only of the mass-centre, in which the forces

are the vertical forces due to gravitation. But although the centre of gravity always and necessarily coincides with the mass-centre, the two terms should be carefully distinguished, as suggesting the distinction between *weight* and *mass*.

§ 216. Since the centre of gravity is that point through which the resultant of the gravitating forces acting on the body always passes, it follows that the body will be supported if acted upon by a force equal to this resultant—that is, to the weight of the body passing through the centre of gravity vertically *upwards*.

For all practical purposes, therefore, the centre of gravity may be described as the point at which the whole weight of the body may be supposed to act.

The centre of gravity is not always within the body, as in a hollow cup, but in order that the body may be supported the applied force must act through that point.

§ 217. **If a heavy body be suspended at any point the Centre of Gravity will be in the vertical line drawn through the point of suspension.**—Let the body be suspended at S. Let W be the weight of the body and G its centre of gravity.

Then W acts at G vertically downwards. The reaction of the point of support is equivalent to a

force acting vertically upwards at S. For equilibrium these two forces must pass through the same point (§ 211, i.). Therefore G must be in a

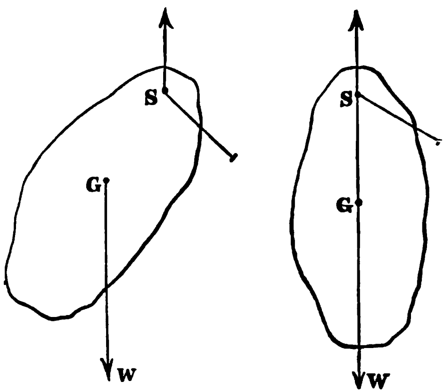

FIG. 106

vertical line with S. In any other position of the body the moment of W about S would cause rotation.

§ 218. **Method of finding the Centre of Gravity of uniform Laminæ.**—The foregoing property enables us experimentally to find the centre of gravity of uniform laminæ or flat bodies of inconsiderable thickness. For if the figure be first suspended at one point, and then at another point, the centre of gravity will be in the verticals through each of these points of suspension; and if these vertical lines be marked on the body

their point of intersection will be the centre of gravity required.

If the centre of gravity be known, we can similarly determine the point in the opposite side of the surface which will be vertically under the

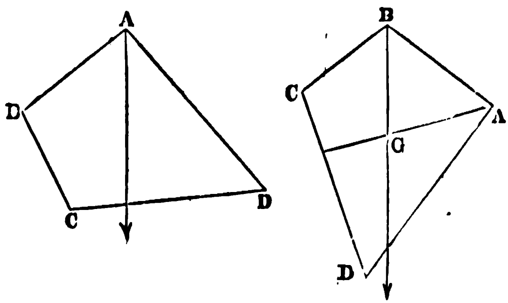

FIG. 107

point of suspension. For if the line joining the centre of gravity with the point of suspension be produced to cut the opposite side, the point of intersection will be the point required.

§ 219. **If a body rest on a hard plane surface, it will stand or fall according as the vertical from the Centre of Gravity falls within or without the base.**—The first condition that a body may rest on a surface is that the surface shall be strong enough to support the body. Otherwise the body under the action of gravity falls through. If the material be sufficiently strong it exerts

a force perpendicular to its surface, which may be considered as acting at some point in the surface in contact with the body. If the vertical through G (fig. 108) fall within the base, the resultant

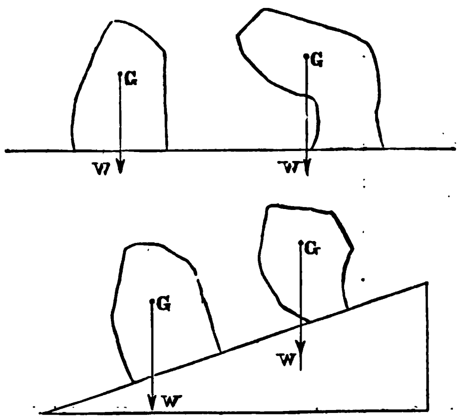

FIG. 108

reaction of the plane will be in the same line with the direction of W, and there will be equilibrium. But if the vertical through G fall beyond the base, W and the reaction cannot be in equilibrium, and the body will fall.

The condition is the same whether a body rests on a horizontal or inclined plane.

When it is said that the vertical from the centre of gravity must fall within the *base*, it must be observed that by the base of a body is

meant the line drawn round the points of support by passing a flexible cord round these points and

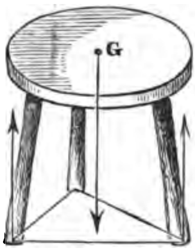

FIG. 109

FIG. 110

stretching it till it is quite tight. For if a body rest on three points of support, as in fig. 109, these

three reactions have a resultant which in equilibrium passes through *G*. If the body rests on more points of support the same holds good. If, however, the vertical from *G* (fig. 110) falls *without* the circumference of the points of support, the resultant of the several reactions cannot pass through *G* and equilibrium cannot exist.

§ 220. **Stable, unstable, and neutral Equilibrium.**—Equilibrium may be of three kinds :—

1. The body may be in such a position that if *slightly* displaced it tends to *return to* its original position, in which case the equilibrium is *stable.*

2. Or, it may tend to *move further away from* its position, in which case the equilibrium is *unstable.*

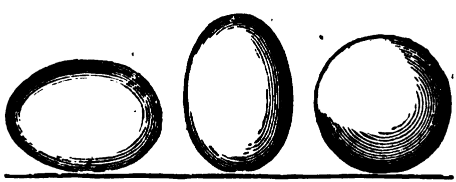

FIG. 111

3. Or, it may *remain in* its new position, in which case the equilibrium is *neutral.*

These conditions are illustrated when a body rests on a hard surface or when suspended by a smooth peg. An egg on either end is in unstable equilibrium, when resting on a longer side it is in

stable equilibrium in one direction, in neutral equilibrium in another. A sphere or cylindrical roller resting on a horizontal surface is in neutral equilibrium. A disc suspended at its centre of gravity will rest in neutral equilibrium. If suspended at

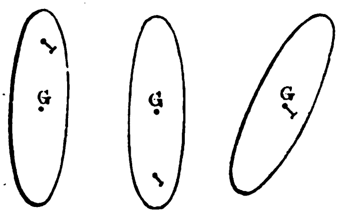

FIG. 112

any other point there are two positions *only* in which it will rest. In the one it fulfils the conditions of stable, in the other of unstable equilibrium. These positions are shown in figs. 111 and 112.

It will be observed that a body when displaced

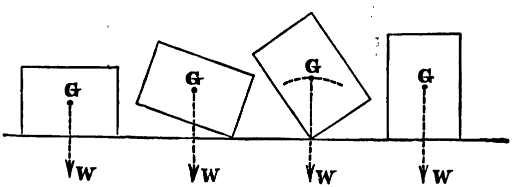

FIG. 113

assumes a position of unstable equilibrium in passing from one position of stable equilibrium into another; and that bodies having several surfaces on which they can rest have positions of

stable and unstable equilibrium corresponding to each. Hence there are some positions in which the equilibrium is more stable than in others.

§ 221. **Position of Centre of Gravity in stable and unstable Equilibrium.**—When a body is in stable equilibrium it will, if slightly displaced, return to its original position, and the centre of gravity of the body will describe an arc, first rising and then falling. If the body be further displaced the centre of gravity will gradually rise till it has reached its highest position, in which case the equilibrium will be unstable.

Hence, in stable equilibrium the centre of gravity occupies the lowest possible position; and

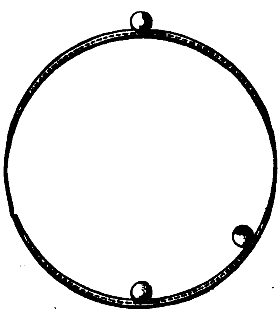

FIG. 114

in unstable it occupies its highest position; and therefore when a body is at rest, its centre of gravity must occupy its highest or lowest position. This may be illustrated by taking a flat wooden hoop

(fig. 114) on the inside and outside of which a groove is cut capable of holding a marble. If the hoop be stood in a vertical position it will readily be seen that there are two positions, and two only, in which the marble can rest. It may, with very great care, be made to rest in the outside groove on the summit, and the slightest displacement will cause it to roll down; or it may remain in stable equilibrium in the inside groove at the bottom of the hoop. In any other position equilibrium is impossible.

§ 222. A body seemingly in unstable equilibrium may really be in stable equilibrium if, by the addi-

FIG. 115

tion of weights, the centre of gravity is brought under the point of support. Thus a half-crown piece will not rest with its edge on the rim of a

wine-glass, but if two silver forks hang by their prongs to the coin it will be found to be supported in stable equilibrium. In the toy represented in the annexed diagram (fig. 115) the centre of gravity g is under the point of support, and the figure consequently can undergo some amount of displacement in certain directions without the equilibrium being destroyed.

§ 223. **The energy of a body in its three states of Equilibrium.**—Bearing in mind the distinction between kinetic and potential energy (§ 107), we shall readily see that the potential energy, or energy of position, which a body possesses varies with its condition of equilibrium. In neutral equilibrium, the potential energy is the same for all positions of the body. In stable equilibrium the potential energy is a minimum—in other words, the body is in the most unfavourable position for doing work—whilst in unstable equilibrium the potential energy is a maximum, and the application of the smallest force can at once convert a part of this energy of position into energy of motion. A body in unstable equilibrium may be said, therefore, so far as its position is concerned, to possess a maximum amount of potential energy.

§ 224. A body which is prevented from sliding is on the point of overturning when the vertical from its centre of gravity falls on the line defining

the base of the body, in which case the body is in a position of unstable equilibrium for the particular direction of motion in which the body is being turned. If a body rest on a rough plane, and the plane be inclined through such an angle that the vertical from the centre of gravity falls at the edge of its base, any further elevation of the plane will cause the body to overturn, provided the plane is sufficiently rough to prevent sliding.

§ 225. **Example.**—To find the inclination of a plane at which overturning takes place. Let G be the

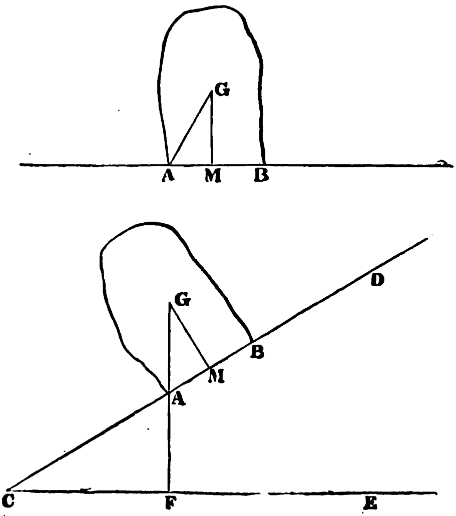

FIG. 116

centre of gravity of a body resting on a rough inclined plane $C\,D$, in unstable equilibrium. Then if $A\,B$ be

the base of the body, the vertical from G passes through A, and is at right angles to the horizontal line CE drawn through C. If GM be drawn perpendicular to AB, the right-angled triangles AGM, ACF will have the angle GAM in the one equal to the angle CAF in the other, and consequently the remaining angle, AGM, equal to the angle ACF. But ACF is the angle of the plane, and therefore the angle at which the plane may be inclined must be less than the angle contained by the lines GA and GM, in order that the body may be in equilibrium. Or, the body will be on the point of overturning when the angle of the plane is such that

$$\frac{\text{the height of the plane}}{\text{the base of the plane}} = \frac{AM}{GM}.$$

XXVIII. *Methods of finding the Mass-Centre, or Centre of Gravity, of a System of Particles or Bodies*

Different methods have to be adopted for finding the mass-centre, or centre of gravity, of bodies, according to their form. All these methods, however, are mostly modifications of the problems of finding the resultant of a number of parallel forces acting at known points. We commence with a system of particles.

§ 226. **To find the mass-centre of two particles.**—Let A and B be the positions of the two particles, whose masses are m_1 and m_2.

Suppose these two particles to be acted upon by parallel forces, such as gravitation, proportional to their masses, then the resultant of these two forces will be at a point G such that

$$\frac{m_1}{m_2} = \frac{G\,B}{A\,G},$$

$$\text{or } m_1 \times A\,G = m_2 \times G\,B,$$

and G is the mass-centre, or, if we consider the

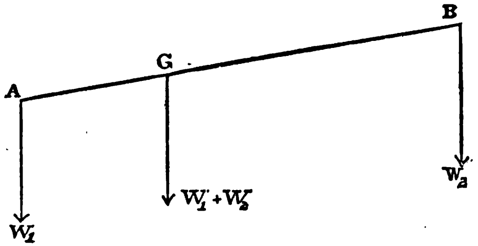

FIG. 117

weights of the particles, the centre of gravity of the two particles.

If W_1 and W_2 be the weights of the two particles the whole weight $W_1 + W_2$ may be supposed to act at g, and the whole mass $m_1 + m_2$ may be considered as at that point.

§ 227. **Mass-Centre of Homogeneous and Symmetrical Bodies.**—In finding the mass-centre of different bodies we shall suppose, unless otherwise stated, that the matter of which they are composed is uniformly distributed throughout their bulk, so

that the masses of parts of the body will be proportional to their volumes. When we speak of the mass-centre of a line we suppose the line to be of uniform section and density; and when we speak of the mass-centre of a surface we suppose the surface to be a thin *lamina* or plate, the thickness and density of which, being uniform, can be neglected. Hence for the masses of thin uniform laminæ we may substitute the areas of their surfaces.

If a body be symmetrical about a plane, or if a surface be symmetrical about a line, the mass-centre of the body is in that plane or line, and coincides with the geometrical centre of all figures that have such a point. Hence—

1. The mass-centre of a straight line is its middle point.
2. The mass-centre of a circle or of its circumference, or of a sphere or of its surface, is its centre.
3. The mass-centre of a parallelogram or of its perimeter is the point in which the diagonals intersect.
4. The mass-centre of a cylinder or of its surface is the middle of its axis.

§ 228. **To find the Mass-Centre of the Surface of a Triangle.**—Let $A\,B\,C$ (fig. 118) be the triangle. Draw the *median* line $A\,D$ from the point A to the centre of $B\,C$, and divide it into a certain number

of equal parts. Through the points of division D', D'', D''' . . . draw parallels to the base BC, and through the points B', C', B'', C'' . . . where the parallels cut the other sides, draw lines parallel to AD. We thus form a series of parallelograms inscribed in the triangle. The line AD passes through the centres O, O', O'' . . . of all these

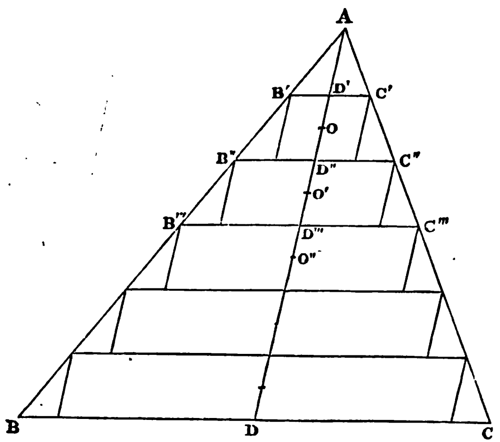

FIG. 118

parallelograms, since it bisects their opposite sides. The mass-centres of these parallelograms are at the points O, O', O'' . . . ; and hence the mass-centre of the sum of these parallelograms is in the line OD. Although the sum of the parallelograms is less than the area of the triangle, it can be shown, by reasoning similar to that employed in § 21, to

approach as near to the area as ever we please, by increasing the number of divisions. Hence the mass-centre of gravity of the triangle is in $A\,D$.

In the same way it may be shown that the mass-centre of the triangle is in the median $C\,E$

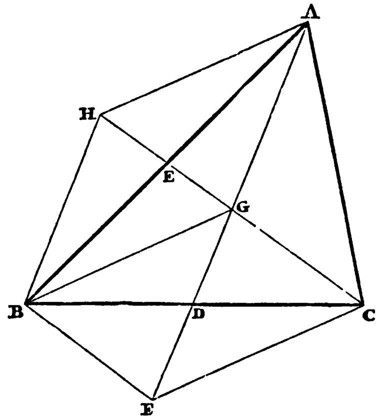

FIG. 119

(fig. 119). Hence the mass-centre of the triangle is at G, where the two medians intersect.

Produce $A\,D$ to F, making $D\,F$ equal to $D\,G$. Produce $G\,E$ to H, making $E\,H$ equal to $G\,E$. Join $B\,G$, $B\,F$, and $F\,C$; join $B\,H$ and $H\,A$.

Then, since the lines $B\,C$ and $G\,F$ are bisected at D, the figure $B\,G\,C\,F$ is a parallelogram, and $B\,F$ is parallel to $C\,G$. For the same reason the figure $H\,A\,G\,B$ is a parallelogram, and $H\,B$ is parallel

and equal to AG. Since HB is parallel to AF, or AG produced, and BF is parallel to HG, the figure HF is a parallelogram;

$$\therefore GF = HB = AG,$$
$$\text{and } \therefore 2\,GD = AG,$$
$$3\,GD = AD,$$
$$\therefore GD = \tfrac{1}{3}AD.$$

Or the mass-centre of a triangle is situated on a median line, and at a distance of $\frac{1}{3}$ of its length from the base.

§ 229. **Example.**—If equal masses m be placed at the angular points A, B, and C of a triangle, the mass-centre of these masses coincides with the mass-centre of the triangle.

The two equal masses at B and C are equivalent to a mass $2\,m$ at D, and the mass-centre of $2\,m$ at D and of m at A will be at G, where AG equal $2\,DG$.

§ 230. **To find the Mass-Centre of the Perimeter of a Triangle.**

The middle points A', B', C' (fig. 120) are the mass-centres of the three lines BC, CA, AB. Join $B'C'$ and divide it in D, so that

$$\frac{B'D}{C'D} = \frac{AB}{AC}.$$

Then D is the mass-centre of the two sides AB, AC.

Join DA', and divide DA' in G, so that

$$\frac{DG}{GA'} = \frac{BC}{AB + AC}.$$

Then G is the mass-centre of the perimeter.

Since $\frac{B' D}{C' D} = \frac{A B}{A C}$ and $A' B' = \frac{1}{2} A B$, and $A' C' = \frac{1}{2} A C$, $\therefore \frac{B' D}{C' D} = \frac{A' B'}{A' C'}$, and, therefore, the line $A' D$ is known (Euclid, vi. 3) to bisect the angle $B' A' C'$. In the same way it can be shown

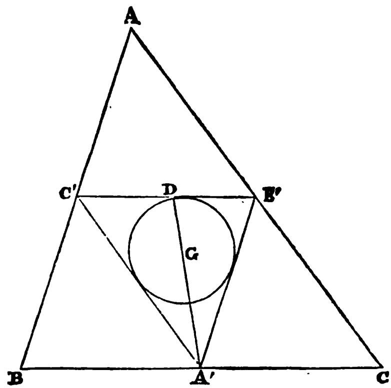

FIG. 120

that G is situated on a line that bisects the angle $C' B' A'$. Hence the mass-centre of the perimeter of the triangle is situated at the centre of the circle inscribed in the triangle $A' B' C'$.

§ 231. **To find the Mass-Centre of any rectilinear figure.**—When a rectilinear figure can easily be divided into a number of triangles, the

mass-centre of the figure can very frequently be found by the following method : Find the mass-centre of each triangle. Then, as the mass of each triangle may be considered as placed at its mass-centre, the mass-centre of these figures, as found by the method given in § 79, will be the point required.

Thus, suppose $A\,B\,C\,D\,E$ a rectilinear figure. Divide it into three triangles by lines drawn

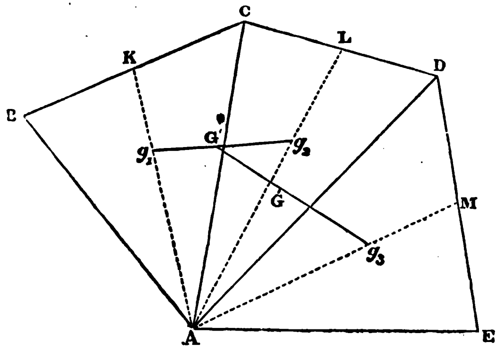

FIG. 121

through A. Draw the medians $A\,K$, $A\,L$, $A\,M$. Then the mass-centre of the triangle $A\,B\,C$ is at g_1, of the triangle $A\,C\,D$ at g_2, and of the triangle $A\,D\,E$ at g_3. Join g_1, g_2, and divide the line in G', so that

$$\Delta\, A\,B\,C \times g_1\,G' = \Delta\, A\,C\,D \times G'\,g_2.$$

Join $G'\,g_3$ and divide this line in G, so that

$$(\Delta\, A\,B\,C + \Delta\, A\,C\,D) \times G'\,G = \Delta\, A\,D\,E \times G\,g_3.$$

Then G is the position of the mass-centre required.

This method is theoretically possible; but in many cases the mass-centre can be found by a method involving fewer mathematical difficulties.

§ 282. **Examples.**—(1) To find the mass-centre of half a regular hexagon.—Let $DABC$ be half a regular hexagon. Then the sides DA, AB, BC are all equal, and if O bisects DC, and OA and OB are joined, the figure is divided into three equilateral triangles.

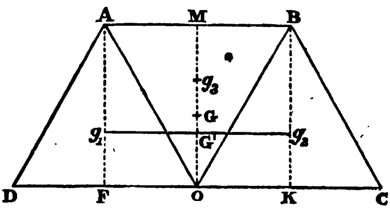

FIG. 122

Draw the medians AF, OM, BK. These are parallel to one another, and perpendicular to DC or AB.

The mass-centre of ΔAOD is at g_1, where $Ag_1 = 2Fg_1$; the mass-centre of ΔBOC is at g_2, where $Bg_2 = 2Kg_2$.

Join $g_1 g_2$. Then, since these triangles are equal their mass-centre is at a point G' which bisects the line $g_1 g_2$. This point, therefore, is on the line OM, and $OG' = \frac{1}{3} OM$. Now, the mass-centre of the ΔAOB is at g_3, where $Mg_3 = \frac{1}{3} OM$, and therefore $Mg_3 = OG' = G'g_3$. Also $\Delta ADO + \Delta BOC = 2\Delta AOB$. Divide $G'g_3$ in G so that

$$2 \triangle A O B \times G' G = \triangle A O B \times G g_3,$$

$$\therefore G' G = \tfrac{1}{3} G g_3 = \tfrac{1}{3} G' g = \tfrac{1}{9} O M,$$

$$\therefore O G = (\tfrac{1}{3} + \tfrac{1}{9}) O M = \tfrac{4}{9} O M,$$

where G is the mass-centre of the half-hexagon.

(2) To find the mass-centre of a figure which consists of half a regular hexagon, having an isosceles right-angled triangle on its longest side.

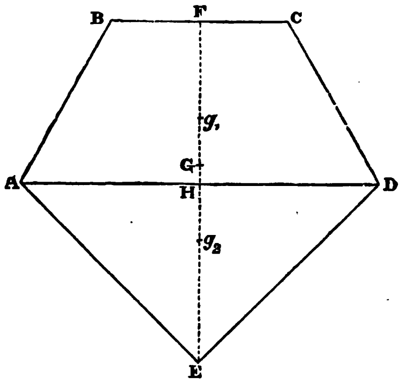

FIG. 123

Let $A B C D E$ be the figure which is symmetrical about the line $E F$. The mass-centre is therefore somewhere in the line $E F$.

Let $a = A B = B C = C D$.

Then $A D = 2a$ and $H E = a$.

Hence the area of the half-hexagon is $3 \dfrac{a \cdot F H}{2}$
$= \dfrac{3 a^2 \sqrt{3}}{4}$,

The area of triangle $A\,D\,E$ equals $\dfrac{A\,D \times H\,E}{2}$ $= a^2$.

Let g_1 be the mass-centre of the half-hexagon, then $H\,g_1 = \frac{4}{9}\,F\,H$ (Ex. 1).

Let g_2 be the mass-centre of the triangle, then $H\,g_2 = \frac{1}{3}\,H\,E$.

Divide $g_1\,g_2$ in G, so that

$$\frac{3\,a^2\sqrt{3}}{4} \times g_1\,G = a^2 \times G\,g_2 \quad . \; . \; . \; (1)$$

Then G is the mass-centre required. Let $g_1\,g_2 = b$ and $g_1\,G = x$; then from equation (1)

$$\frac{3\sqrt{3}}{4}\,x = (b - x),$$

$$\therefore x = \frac{4\,b}{4 + 3\sqrt{3}},$$

where

$$b = \frac{4}{9}FH + \frac{1}{3}HE = \left(\frac{4}{9}\,\frac{\sqrt{3}}{2} + \frac{1}{3}\right)a = \frac{2\sqrt{3} + 3}{9}\,.\,a.$$

Hence $G\,g_1 = \dfrac{4}{9}\,.\,\dfrac{2\sqrt{3} + 3}{3\sqrt{3} + 4}\,.\,a$, which gives the distance of the mass-centre from g_1.

To find the distance of G from H, we can subtract $G\,g_1$ from $H\,g_1$, which gives

$$H\,G = \frac{2}{33}\,(3\sqrt{3} - 4)\,a.$$

EXERCISES XXIV

1. If a parallelogram be suspended at a point in one of its sides, what point will be vertically under the point of suspension?
2. A circular tower, the diameter of which is 20 feet, is

being built, and for every foot it rises it inclines 1 inch from the vertical; what is the greatest height it can reach without falling?

3. A circular table weighs 20 lbs. and rests on four legs in its circumference forming a square; find the least pressure that must be applied at its edge to overturn it.
4. Two masses weighing 17 lbs. and 13 lbs. are connected together by a weightless rod; at what point must the rod be supported to rest in any position whatever?
5. A solid cube rests on a rough plane; through what angle may the plane be inclined before the cube overturns?
6. An equilateral triangle stands vertically on a rough plane; find the ratio of the height to the base of the plane when the triangle is on the point of overturning.
7. A triangular board weighing 30 lbs. is carried by 3 men, each standing at one of the corners; what weight does each bear?
8. Four masses of 3 lbs., 4 lbs., 5 lbs., and 6 lbs. are placed at the corners of a square; find the position of their mass-centre.
9. To one corner of a heavy square a mass of equal weight is attached; where must it be suspended by a single string to rest horizontal?
10. Find the mass-centre of a triangular lamina, to two corners of which masses, respectively equal to half the mass of the triangle, are attached.
11. A ladder 20 feet long weighs 60 lbs.; its centre of gravity is 8 feet from the thicker end; it is carried by 2 men, one of whom supports the heavier end on his shoulder; where must the other stand that the weight may be equally divided?
12. Masses of 1 gram, 2 grams, 3 grams are placed at the corners of an equilateral triangle; find their mass-centre.

13. An isosceles triangle rests on a square, and the height of the triangle is equal to a side of the square; find the mass-centre of the figure thus formed.
14. Find the mass-centre of a figure in the shape of a kite, formed by two isosceles triangles put base to base. The extreme width of the kite is 4 feet, and the heights of the isosceles triangles are 2 feet and 4 feet respectively.
15. Find the inclination of a rough plane on which half a regular hexagon can just rest in a vertical position without overturning, with the smaller of its parallel sides in contact with the plane.

XXIX. *Methods of finding the Mass-Centre of Bodies joined together and of Parts of Bodies*

§ 233. **Given the Mass-Centre of each of two bodies, to find that of both together.**—The mass-centre of both bodies can be found by joining the mass-centres of the two, and dividing the line thus drawn into two parts in the inverse ratio of the masses of the bodies. The process is the same as in finding the mass-centre of two particles.

§ 234. **Examples.**—(1) Two cylinders with circular bases have a common axis; the radius of the one is 4 inches and the altitude 6 inches; the radius of the other is 3 inches and the altitude 8 inches; find the mass-centre of both.

Since the volume of a cylinder is equal to $\pi r^2 h$, where r is radius of the base, and h the height, the

volume of the one cylinder is $\pi \times 16 \times 6$, and of the other $\pi \times 9 \times 8$ cubic inches.

The distance $g_1 g_2 = 7$ inches. If G be the mass-centre required

$$\frac{g_1 G}{g_1 g_2 - g_1 G} = \frac{72\pi}{96\pi};$$

$$\therefore g_1 G = \frac{3}{7} g_1 g_2;$$

$$\therefore g_1 G = 3 \text{ inches};$$

or their mass-centre is at the point in the axis where they touch.

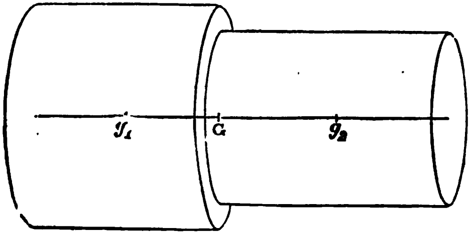

FIG. 124

The mass-centre of several bodies joined together can be found in the same way when their several mass-centres lie on the same straight line; otherwise the problem generally presents numerical difficulties.

(2) To find the mass-centre of three uniform rods joined together so as to form the letter F, the lengths of the rods being as 1 : 3 : 6, and their thickness neglected.

Let the length of the vertical rod be 6 inches, the horizontal rod 3 inches, and the small central rod 1 inch.

Take $g_1 g_2$ and divide it in G', in the ratio of 1 : 2. Then the mass-centre of these two rods is at G'. Draw a vertical through G'. This bisects the central rod

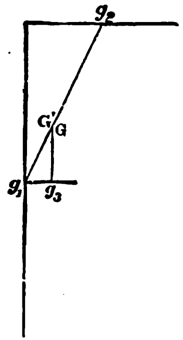

FIG. 125

and passes, therefore, through its mass-centre. Mark off $\frac{1}{10}$ of $g_3 G'$ from G', and the point G thus determined is the mass-centre required.

§ 235. **Given the Mass-Centre of a whole body and of a part of it cut off, to find that of the remainder.**—Let M be the mass of the whole

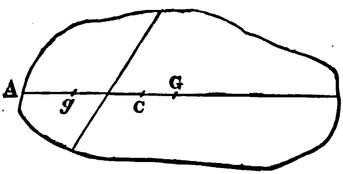

FIG. 126

figure and C its mass-centre. Let m be the mass of a part, and g its mass-centre. Then $M - m$ is the mass of the remainder.

Join $g\,C$ and let the mass-centre of the remainder be at some point G in the straight line $g\,C$ produced.

Produce $C\,g$ to some fixed point A. Then we can find the distance of G from A thus:

$$(M - m)\,C\,G = m\,.\,g\,C \text{ (§ 226)};$$

$$\therefore (M - m)(A\,G - A\,C) = m\,(A\,C - A\,g);$$

$$\therefore A\,G = \frac{M \times A\,C - m \times A\,g}{M - m}.$$

§ 236. **Example.**—To find the mass-centre of the remainder of a square, out of which one of the triangles formed by the diagonals has been taken.

The figure that remains is symmetrical about the line $L\,M$. The mass-centre is, therefore, in that line.

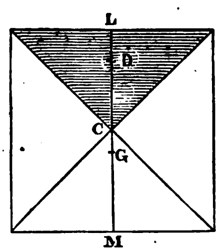

FIG. 127

Let the side of the square be a. Then the area of the whole square is a^2, and its mass-centre is at C; the area of the triangle is $\frac{a^2}{4}$, and its mass-centre is at D; and the area of the remainder is $\frac{3\,a^2}{4}$, and let its

mass-centre be at G, the position of which is required to be found.

Then, since C is the mass-centre of the whole square, and D and G of the parts $\frac{a^2}{4}$ and $\frac{3a^2}{4}$ respectively, we have

$$\frac{a^2}{4} \times DC = \frac{3a^2}{4} \times GC,$$

$$\therefore \frac{a^2}{4} \times \frac{a}{3} = \frac{3a^2}{4} \times GC,$$

$$\therefore GC = \frac{1}{9}a,$$

or G is situated $\frac{1}{9}$ of a from C.

§ 237. General Method of finding the Mass-Centre of a number of particles or bodies in the same plane.

Let $m_1, m_2, m_3 \ldots$ be the masses of the several particles or bodies; $x_1, x_2, x_3 \ldots$ the distances of their mass-centres from a fixed line, in their plane, Oy; and $y_1, y_2, y_3 \ldots$ the corresponding distances from a line Ox at right angles to Oy. Then if X and Y be the respective distances of their mass-centre from these lines

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots},$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}.$$

Let A and B be the positions of the mass-centres of m_1 and m_2 respectively, and let their

common mass-centre be at G_1. Draw $A\,a$, $G_1\,g_1$, and $B\,b$ parallel to $O\,y$.

Then, as before (§ 226),

$$m_1 \times a\,G_1 = m_2 \times G_1\,B,$$

or
$$m_1 \times a\,g_1 = m_2 \times g_1\,b\,;$$

$$\therefore\ m_1 \times (O\,g_1 - O\,a) = m_2\,(O\,b - O\,g_1),$$

$$\therefore\ O\,g_1 = \frac{m_1 \times O\,a + m_2 \times O\,b}{m_1 + m_2}.$$

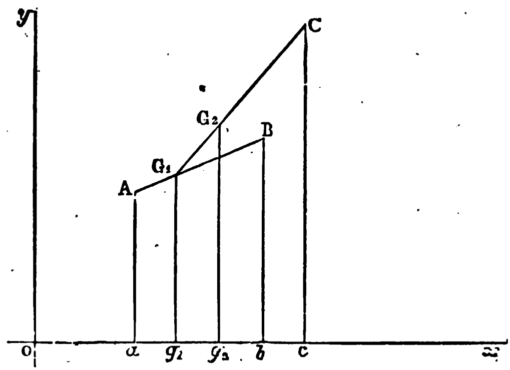

FIG. 128

Let C be the position of the mass-centre of m_3. Join $G_1\,C$, and let G_2 be the position of the mass-centre of $(m_1 + m_2)$ at G_1 and of m_3 at C. Draw lines through C and G_2 parallel to $O\,y$, meeting $O\,x$ in c and g_2. Then

$$(m_1 + m_2) \times G_1\,G_2 = m_3 \times G_1\,C,$$

or $\quad (m_1 + m_2)\, g_1 g_2 = m_3 \times g_1 c$, whence, as above,

$$O g_2 = \frac{(m_1 + m_2) \times O g_1 + m_3 \times O c}{m_1 + m_2 + m_3}$$

$$= \frac{m \times O a + m_2 \times O b + m_3 \times O c}{m_1 + m_2 + m_3}$$

In the same way it may be shown for any number of particles or bodies whose masses and mass-centres are known, that

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ . \ . \ .}{m_1 + m_2 + m_3}$$

where X is the distance of their common mass-centre from $O y$.

If we draw lines from A, B, and C parallel to $O x$, it may be similarly proved that

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ . \ . \ .}{m_1 + m_2 + m_3 + \ . \ . \ .}$$

where Y is the distance of the mass-centre of the whole system from the line $O x$.

By aid of this proposition, which is the most general we have proved, the mass-centre of several particles or bodies, whose mass-centres lie in the same plane although not in the same straight line, may be determined.

§ 238. The product $m x$ may be called the *mass-moment* of the mass m about any line the perpendicular distance of which from its mass-centre is x. We see, therefore, that the mass-

moment of several masses is equal to the sum of their several mass-moments about any line in their plane.

§ 239. If the particles are not in the same plane, and if x_1, x_2, x_3, and $y_1, y_2, y_3 \ldots$ be their distances from two fixed planes at right angles to each other, the values of X and Y already found can be proved to be true by a precisely similar process of reasoning. Also if $z_1, z_2, z_3 \ldots$ be the distances of their mass-centres from a third plane at right angles to each of the other two, we can similarly show that

$$Z = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}, \text{ where}$$

Z is the distance of their mass-centre from this plane.

§ 240. **Example.**—From the point D (fig. 129) which bisects $A\,C$, one of the sides of the equilateral triangle $A\,B\,C$, a straight line $D\,E$ is drawn perpendicular to the base cutting off the triangle $D\,E\,C$; required the mass-centre of the figure $A\,D\,E\,B$.

Take $B\,C$ as one fixed line, and $B\,C'$ at right angles to it; find g_1, the mass-centre of the whole triangle, and g_2 of the part $D\,E\,C$. Let $B\,F = X$, $B\,M = x_1$, and $B\,K = x_2$, $g_1\,F = Y$, $G\,M = y_1$, and $g_2\,K = y_2$.

The area of the whole triangle equals $m_1 + m_2 = \frac{a\,h}{2}$, where $B\,C = a$ and $A\,F = h$; the area of the

small triangle equals $\frac{a\,h}{16}$; the area of the remaining figure equals $\frac{7}{16}\,a\,h$.

Then, if G be the position of the mass-centre of the figure $A\,D\,E\,B$, we have

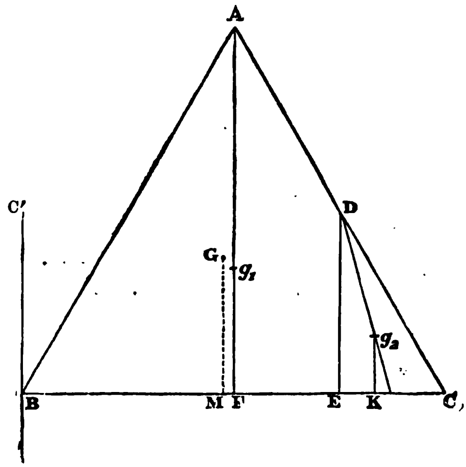

FIG. 129

$$(m_1 + m_2)\,X = m_1\,x_1 + m_2\,x_2\,;$$

or $$\frac{a\,h}{2} \times \frac{a}{2} = \frac{7}{16}\,a\,h\,.\,x_1 + \frac{a\,h}{16}\,.\,\frac{5}{6}\,a.$$

Since $$B\,F = \frac{a}{2} \text{ and } B\,K = \frac{5}{6}\,a,$$

$$\therefore\; 24\,a = 42\,x_1 + 5\,a,$$

and $$x_1 = B\,M = \frac{19}{42}\,a.$$

Similarly $$(m_1 + m_2)\,Y = m_1\,y_1 + m_2\,y_2,$$

or $$\frac{a\,h}{2} \times \frac{h}{3} = \frac{7}{16}\,a\,h\,.\,y_1 + \frac{a\,h}{16}\,.\,\frac{h}{6}.$$

Since $$g_1 F = \frac{1}{3} h \text{ and } g_2 K = \frac{1}{6} h,$$

$$\therefore y_1 = G M = \frac{15}{42} h.$$

Hence the mass-centre of the figure $A\,D\,E\,B$ is at a point G, such that $G\,M = \frac{15}{42}\,h$, and $B\,M = \frac{19}{42}\,a$.

§ 241. **To find the Mass-Centre of the Surface of a Cone.**—If two points a and b be taken in

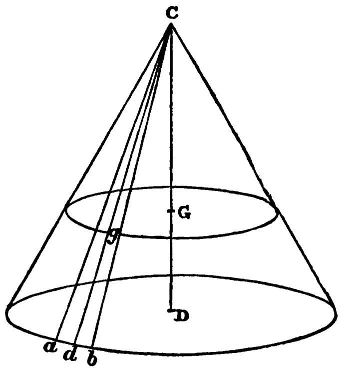

FIG. 130

the circumference of the base of the cone, and each of these points be joined with C, the vertex, the figure thus formed may be regarded as a triangle if the points a and b be sufficiently near one another. The mass-centre of this triangle is at a point g, one-third up the median line

C d. Now, the circumference of the base may be regarded as a polygon with an infinite number of infinitely small sides; and the surface of the cone may therefore be supposed to consist of an infinite number of such triangles, as *C a b.* If a circle be drawn through *g* parallel to the base, the mass-centre of each of these triangles will lie on the circumference of this circle, and therefore the centre of this circle will be the mass-centre of the sum of these triangles, i.e. of the surface of the cone. But the centre of this circle is at *G*, where *D G* is equal to one-third of *D C*; therefore the mass-centre of the surface of a cone is at a point in the axis at a distance of *one-third* of the axis from the centre of the base.

§ 242. **Mass-Centre of a Pyramid.**—Since a pyramid may be supposed to consist of a number of parallel discs or flat prisms of the same shape as the base, lying one on the other, the mass-centre of all these discs, and therefore of the pyramid, must be situated in the line joining the mass-centre of the base with the apex of the pyramid, and it can easily be shown to be at a distance of *one-fourth* of this line from the base.

Thus, if *C* be the mass-centre of the base of a pyramid, and *A* be its apex, and if *A C* be joined, and *C G* be marked off equal to one-

fourth of CA, then G is the mass-centre of the pyramid.

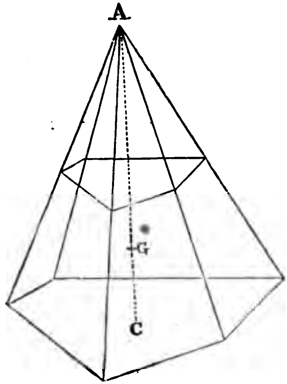

FIG. 131

§ 243. **Mass-Centre of a Cone.**—If the number of sides of a pyramid be increased without limit, the pyramid becomes a cone. The mass-centre of a cone is found, therefore, in the same way as that of a pyramid, by joining the apex of the cone with the mass-centre of its base, and measuring one-quarter of this line from the base.

EXERCISES XXV

1. A heavy wire is bent in the form of a right angle, and one arm is twice the length of the other. At the angular point a mass, equal to half the mass of the whole wire, is fixed; find the mass-centre of the system.

2. A cylinder the diameter of which is 10 feet, and height 60 feet, rests on another cylinder, the diameter of which is 18 feet, and height 6 feet; and their axes coincide; find their common mass-centre.
3. An equilateral triangle rests on a square, and the side of the triangle is equal to the side of the square; find the mass-centre of the figure thus formed.
4. Into a hollow cylindrical vessel 11 inches high, and weighing 10 oz., the centre of gravity of which is 5 inches from the base, a uniform solid cylinder 6 inches long, and weighing 20 oz., is just fitted; find their common centre of gravity.
5. From a uniform circular disc another disc, having for its diameter the radius of the first circle, is cut away; find the mass-centre of the remainder.
6. From a square the sides of which is 6 inches a corner square is cut away, the side of which is 2 inches; find the mass-centre of the remainder.
7. A heavy wire is bent at its middle point, so as to contain an angle of 60°; it is suspended from one of its ends; find its position in equilibrium.
8. $ABCD$ is a square; CB is produced to E, and on EB is described a square. If $BE = \frac{1}{3} AB$, find the mass-centre of the figure thus formed.
9. Find the height of a cylinder which can just rest on a rough inclined plane the angle of which is 60°, the diameter of the cylinder being 6 inches.
10. From a rectangle 8 inches by 5 inches a corner rectangle is cut out 2 inches by 3 inches (2 inches from the side of 8 inches); find the mass-centre of the remainder.
11. A rectangular board 6 inches by 8 inches, and weighing 2 oz., is hung up from one of its angular points; to the extremity of the adjacent short side a body weighing 8 oz. is attached; find the position in which the board hangs in equilibrium.
12. A trapezium, having two parallel sides, which are 4

and 12 feet long, and the other sides each equal to 5 feet, is placed with its plane vertical, and with its shortest side on an inclined plane; find the relation between the height and base of the plane when the trapezium is on the point of falling over.

13. Find the centre of gravity of three equal rods, AB, AC, and AD, in the same plane, and diverging from the point A, each of the angles BAC and CAD being one-third of a right angle.
14. Masses of 2, 3, 2, 6, 9, 6 kilograms are placed at the angular points of a regular hexagon taken in order; determine the position of their mass-centre.
15. A cylindrical vessel weighing 4 lbs., and the internal depth of which is 6 inches, will just hold 2 lbs. of water. If the centre of gravity of the vessel when empty is 3·39 inches from the top, determine the position of the centre of gravity of the vessel and its contents when full of water.
16. Two uniform cylinders of the same material, one of them 8 inches long and 2 inches in diameter, the other 6 inches long and 3 inches in diameter, are joined together, end to end, so that their axes are in the same straight line. Find the mass-centre of the combination.
17. A short circular cylinder of wood has a hemispherical end. When placed with its curved end on a smooth table it rests in any position in which it is placed. Determine the position of its centre of gravity.

EXAMINATION QUESTIONS VIII

1. Masses of 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 oz. are placed along a line at intervals of 3 inches. Find the position of their centre of gravity.—*Univ. Lond. Matric.*, June 1878.
2. A telescope consists of three tubes, each 10 inches in length, sliding within one another, and their weights

are 8, 7, and 6 oz. Find the position of the centre of gravity when the tubes are drawn out to their full length.—*Ib.* Jan. 1879.

3. What is the centre of gravity of a body? A line is drawn across an equilateral triangle, of 12 inches side, parallel to its base, and so as to cut off one-fourth of its area. Find the distance of the base from the centre of gravity of the remainder.—*Ib.* Jan. 1880.

4. A solid right circular cone of homogeneous iron is 64 inches in height, and its mass is 8,192 lbs. The cone is cut by a plane perpendicular to the axis, so that the mass of the small cone removed is 686 lbs. Find the height of the centre of gravity of the truncated portion remaining, above the base of the cone.—*Ib.* June 1880.

5. Show that the centre of gravity of three equal weights placed at the angles of a triangle coincides with the centre of gravity of the triangle.

   $A\,B\,C$ is an equilateral triangle of 6 inches side, of which O is the centre. If the triangle $O\,B\,C$ be removed, find the distance from A to the centre of gravity of the remainder.—*Ib.* Jan. 1881.

6. Weights of 1 lb., 2 lbs., 3 lbs., and 4 lbs. are suspended from a uniform lever 5 feet long at distances of 1 foot, 2 feet, 3 feet, and 4 feet respectively from one end. If the mass of the lever is 4 lbs., find the position of the point about which it will balance.—*Ib.* Jan. 1882.

7. A uniform rod $A\,B$ is 4 feet long and weighs 3 lbs. 1 lb. is then attached to the end A, 2 lbs. at a point distant 1 foot from A, 3 lbs. at 2 feet from A, 4 lbs. at 3 feet from A, and 5 lbs. at the end B. Find the distance from A of the centre of gravity of the system.—*Ib.* June 1882.

8. A uniform plate of metal 10 inches square has a hole 3 inches square cut out of it, the centre of the hole being $2\frac{1}{2}$ inches distant from the centre of the plate.

Find the position of the centre of gravity of the plate.—*Ib.* June 1883.

9. Weights of 1 lb., 2 lbs., 3 lbs., 4 lbs. are placed at the angular points A, B, C, D respectively of a square $A\,B\,C\,D$. Find the distance of the centre of gravity of the system from the centre of the square.—*Ib.* June 1884.

10. A uniform triangular lamina, whose sides are 3, 4, and 5 inches respectively, is suspended by a string from the middle point of the longest side. Draw a figure showing clearly the position of the opposite angular point. If equal weights be attached to the three angular points, how will the position of equilibrium be affected?—*Ib.* Jan. 1886.

11. Five masses, of 1, 2, 3, 4, 5 ounces respectively, are placed upon a square table. Their distances from one edge of the table are 2, 4, 6, 8, 10 inches, and from the adjacent edge 3, 5, 7, 9, 11 inches respectively. Find the distances of their centre of mass (centre of gravity) from the two edges.—*Ib.* Jan. 1888.

12. A rod 12 feet long has a weight of 1 lb. suspended from one end, and when 15 lbs. is suspended from the other end it balances at a point 3 feet from that end, while if 8 lbs. be suspended there it balances at a point 4 feet from that end. Find the weight of the rod and the position of its centre of gravity.—*Ib.* Jan. 1889.

13. A cylinder, whose base is a circle 1 foot in diameter, and whose height is 3 feet, rests on a horizontal plane with its axis vertical. Find how high one edge of the base can be raised without causing the cylinder to turn over.—*Ib.* June 1889.

14. The mass of an equilateral triangle is 4 lbs. Masses of 1, 1 and 2 lbs. respectively are placed at the angular points. Find the centre of mass of the system.—*Ib.* June 1890.

15. Explain fully the circumstances under which a rect-

angular block, standing on a plank which is being gradually tilted, shall topple over, being prevented from sliding by a small obstacle. As an example, take the case of a block $8 \times 5 \times 5$ cubic inches.—*Ib.* June 1891.

16. Find the load which must be placed at one corner of an equilateral triangular plate to bring the centre of gravity to the middle point of a perpendicular bisector (or median line).—*Ib.* Jan. 1892.

17. $A\,B\,C$ is an equilateral triangle, each side being 2 inches long; particles whose masses are 1, 2, 3 are placed at A, B, C respectively; find their centre of gravity by construction, and note its distance from A.—*S. & A. Dep.* 1891.

18. $A\,B\,C\,D$ is a square lamina of uniform density; $E\,F$ are the middle points of $A\,B$ and $B\,C$; if the corner of the square is turned down along the line $E\,F$ (so that B comes on to the diagonal $A\,C$), find the centre of gravity of the lamina under the new circumstances.—*Ib.* 1891.

ANSWERS TO EXERCISES AND EXAMINATION QUESTIONS

EXERCISES I. P. 17.

(1) $122\frac{2}{3}$ ft. per sec. (2) 40 secs. (3) 66 ft. per sec. (4) $166\frac{2}{3}$ cents. per sec. (5) 6,283·2 cents. (6) $8\frac{4}{5}$ ft. per sec. (7) 16 hours. (8) 93·876 metres per min. (9) The velocity of 580 metres per hour, by 10 metres per hour. (10) 7·5 ; 316,800 ; 7,920.

EXERCISES II. Pp. 29–30.

(1) 40 secs. (2) 2·4 ft. (3) 120 ft. per sec. (4) 20 secs. (5) 257·6 ft. (6) $a=20$; vel. $=200$ ft. per sec. (7) 200 ft. per sec. (8) 12,250 cents. (9) $-9\frac{1}{3}$ (cents., secs.) (10) 1,391,040 (ins., mins.) (11) $\frac{65}{432}$ cent. (12) 170 ft. nearly. (13) 240 (miles, hours); $\frac{22}{225}$ (ft., secs.) (14) 195 ft. (15) 40 ft. per sec. (16) ·3 (ft., secs.); 20 yards. (17) 30 ft. per sec. (18) 60 ft. per sec.

EXERCISES III. P. 39.

(1) 32 (ft., secs.) (2) Draw $A\,B$ to represent 5 secs., and draw $A\,C$ and $B\,D$ at right angles to AB to represent the two accelerations. Join $C\,D$. The area of the figure $A\,B\,D\,C$ represents the distance. (3) 66 yards. (4) $\frac{1}{\sqrt{10}}$ sec., $\frac{\sqrt{2}-1}{\sqrt{10}}$ sec., $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}$ sec. (5) $\frac{13}{120}$ (cents., secs.) (6) $a = 6$ (ft., secs.), $u = 7$ ft. per sec.

EXERCISES IV. Pp. 52–5.

(2) 37·36 miles per hour, nearly. (3) No. (4) $10\sqrt{5}$ ft. (5) $16\sqrt{2}$. (6) 595 yards. (7) $117{\cdot}\dot{3}$ secs.; 1,147·2 yards. (8) 10 ft. per sec., along line bisecting angle of 120°. (9) $109\frac{1}{11}$ ft. from A. (10) 20 ins. per sec. (11) $2\frac{3}{11}$ miles from A's starting point. (12) $\frac{11}{75}$ (ft., secs.) (13) 5 (ft., secs.) (15) 50 ft. North. (16) $12\sqrt{13}$ ft. (17) 33 ft. (19) 1,500 ft. (22) 90 ft. (23) $5\sqrt{3}$ ft. per sec. (24) $10\sqrt{2}$.

EXAMINATION QUESTIONS I. Pp. 55–8.

(2) $5\sqrt{3} = 8{\cdot}66$ mins., nearly, at 120° to direction of the stream. (4) 2 (ft., secs.) (5) 55 ft. (6) Distance $= 3\frac{19}{72}$ miles. (7) 5 ft. per sec.; $\sqrt{29} = 5{\cdot}385$ ft. per sec. (8) $5\frac{13}{15}$ (ft., sec.) (9) Take $A\,B$, 4 units in length, in direction opposed to the current, and draw $B\,C$ at right angles to it, meeting in C a circle, centre

A and radius 6 units in length ; $A\,C$ is direction required. (10) $\frac{1}{3}$ sec. ; $\frac{\sqrt{2}-1}{3}$ sec. ; $\frac{\sqrt{3}-\sqrt{2}}{3}$ sec. (11) 1600 ft., 320 ft. per sec. (12) 10 miles an hour, N.W. (13) 22·0593 ft. nearly. (14) 2 ft. per sec., 30° W. of S. (15) 13·6 ft. per sec. (16) 24,000 (yards, mins.) (17) 6 ft. per sec. (18) 66,000 (yards, mins.) (19) 36·3 secs. (20) 4 (ft., secs.) ; 5 secs.

Exercises V. Pp. 67–70.

(1) 100 ft. (2) 160 ft. per sec. (3) 176 ft.; 576 ft. (4) 400 ft. (5) 5 secs. (6) 204 ft. (7) 178 ft. per sec. (8) $16\sqrt{10}$ ft. per sec. (9) 100 ft. (10) 187·44 ft. (11) $67\frac{1}{3}$ ft. (12) 135 ft. (13) 336 ft. (14) 224 ft. (15) 10 ft. per sec. (16) 48 ft.; 144 ft.; 208 ft. (17) 88 ft. (18) $1\frac{7}{8}$ sec., after second body is projected. (19) 96 ft. per sec. (20) 52·8 ft. (21) 96 feet per sec.; 8 secs. (22) 144·9 ft.; 16·1 ft. per sec. *downwards.* (23) 74 ft. per sec. (24) 114 ft.; 144 ft. (25) 160 ft. (26) 140 ft. per sec. *downwards.* (27) 1,608 ft. per sec. (28) $g = 32{\cdot}16$ (ft., secs.) $= 38{,}592$ (yards, mins.) (29) 5 secs.; $247\frac{1}{2}$ metres.

Exercises VI. Pp. 79–80.

(1) $\frac{\sqrt{10}}{2}$ sec. $= 1{\cdot}58$ sec. (2) $8\sqrt{10}$ ft. per sec. $= 25{\cdot}298$ ft. per sec. (3) 38·4 ft. per sec. (4) 95·7 ft. nearly. (5) 824 yards. (6) $68\frac{3}{4}$ secs. (7) 312·5 ft.; 1,250 ft. (8) 64 ft. (9) 36 ft.; 75 ft.; 8 secs. (10) 156 ft. (11) 330 ft., nearly.

Examination Questions II. Pp. 81–3.

(1) 144 ft. (2) 402·5 ft.; 177·1 ft. (3) $174\frac{3}{8}$ ft. per sec.; 784 ft. If the balloon is still ascending when the stone is let fall, $v = 68{\cdot}17$ ft. per sec; $s = 306{\cdot}76$ ft. (4) § 41. (5) Three-quarters of the way up. (6) 2 secs. (7) 225 ft.; $2\frac{1}{2}$ or 5 secs. (8) East. (9) 8·05 ft.; 120·75 ft., down the plane. (10) 864 ft.; 464 ft. per sec. (11) $4\frac{2}{3}$ ft.; 0·16 ft. (12) 136 ft. per sec.; 88 ft. (13) 80 ft. (14) 64 ft. (15) See § 45. (16) 16 ft. per sec.; 32 ft. per sec.; § 14. (17) 1,200 ft. per sec. (18) $112\frac{1}{2}$ ft.; $8\frac{3}{4}$ secs. (19) 127·421 metres. (20) 5·48 secs. nearly; 87·68 ft. per sec. nearly. (21) $65\frac{5}{48}$ ft.; $5\frac{5}{24}$ secs. (22) 490·5; 30°. (23) 350 ft. (24) 173·2 ft. per sec. nearly. (25) $3\frac{1}{2}$ secs. after first stone is thrown; at height 84 ft.; the first stone will be descending, the second ascending. (26) $\frac{1}{8}$ length of plane. (27) $\sqrt{\frac{h}{g}}\ (1 + \sqrt{3})$ secs. (28) 1:5. (29) 172 ft. per sec. (30) a, 3·2; b, 25·29 ft. per sec.; c, 7·9 secs.

Exercises VII. Pp. 112–6.

(1) 50,000 dynes. (2) 32 (ft., secs.) (3) 1,564·62 grams weight. (4) 2 : 7. (5) 9,800 cents.; 980 cents. per sec. (6) 321 : 314. (7) 1,200 g. (8) 6 ft. per sec. (9) 3 secs. (10) 3 secs. (11) 9 : 1. (12) 210 poundals. (13) $\frac{1}{20}$ approximately. (14) 10·125. (15) $\frac{1}{2}$ pound. (16) 3 ft. (17) 7·2 oz. (18) 6 lbs. weight. (19) $\frac{1}{3}$ oz. (20) 5 lbs. weight. (21) 3·75 lbs. weight. (22) $5\frac{13}{17}$ oz. (23) $\frac{9}{64}$ ft.; 268 ft. (24) 250 poundals. (25) 15 ft. $2\frac{1}{4}$ ins. (26) 625 secs. (27) 125

ft. (28) 2 secs. (29) $11\frac{33}{52}$ oz.; $14\frac{37}{52}$ oz. (30) $4\frac{8}{9}$ mins. (31) $13\frac{3}{4}$ lbs. weight. (32) up, 126 lbs. weight; down, 98 lbs. weight. (33) $16\sqrt{2}$ ft. (34) 10 ft. per sec. (35) 1 minute. (36) 137·5 secs.; 10 miles per hour. (37) $\frac{16}{3}\sqrt{3}$ ft. per sec. (38) 25,804,800. (39) $\frac{1}{2}g$ down plane at 30°; $\frac{\sqrt{3}}{2}.g$ down plane at 60°. (40) $5\frac{184}{187}$ secs.

Exercises VIII. Pp. 121–2.

(1) 180 ft. per sec. (2) $266\frac{2}{3}$ ft. (3) 3 : 2. (4) 7 units of momentum. (5) 9 : 4. (6) 1,500,000 *poundems.* (7) 7,200 poundals, or 225 lbs. weight. (8) 32,000 poundals = 1,000 lbs. weight.

Examination Questions III. Pp. 123–6.

(1) 5 : 1. (3) 2 : 3. (4) 40 miles per hour; $\frac{4}{5}$ mile. (5) 11 : 1,080. (7) 2 tons 7 cwt. (8) 181·125 ft. (11) $3\frac{3}{4}$ oz.; $4\frac{1}{4}$ oz. (12) 980. (13) In the one the stone will seem to remain stationary; in the other to fall freely, as if the chamber were at rest. (14) 492 grams. (15) 24 lbs. 10 oz. (16) $\frac{16}{5}$ (ft., secs.); 40 ft. (17) 3 ft. per sec.; $4\frac{1}{2}$ ft. (18) This 'tension' or pull equals the weight of $\frac{1}{2}$ lb.; forces pressing on each scale pan are $6\frac{2}{3}$ oz., $5\frac{1}{3}$ oz. (19) $6\frac{1}{4}$ lbs. weight; $\frac{5}{56}$ (ft., secs.) (20) 13 lbs. weight; 256·12 ft. per sec. (21) $6\frac{2}{3}$ (ft., secs.) (22) 11 : 1,280. (23) $7\frac{1}{16}$ lbs. weight; $237\frac{3}{8}$ lbs. weight. (24) 3,200. (25) 440 ft. (26) 325 lbs. $8\frac{1}{3}$ oz. nearly. (27) 14 cents. per sec. nearly. (28) More at the bottom. (29) 7 : 8. (30) $M : m :: 15 : 2$.

Exercises IX. P. 144.

(1) $75\frac{5}{13}$ (cents., secs.); $129\frac{3}{13}$ grams weight. (3) 12·8 (ft., secs.) (4) 715·5 ft. per sec. (5) 3 : 2. (6) 12·5 ft. per sec. (7) 11 : 1. (8) 4 ft. per sec. (9) $\frac{v}{2}$; $\frac{v}{3}$; $\frac{v}{4}$. (10) $3\frac{4}{7}$ ft. per sec. (11) 2 ft. (12) 10 ; $6\frac{2}{3}$; 5 ; 4 ; $3\frac{1}{3}$ and so on. (13) (1) The first goes on with velocity 151·25 cents. per sec., and the second with velocity 131·25 cents. per sec. (2) The first rebounds with velocity 161·25 cents. per sec., and the second with velocity 18·75 cents. per sec. (14) 75 cents. per sec. ; 150 cents. per sec. (15) $v = -eu$. (16) $e = \frac{3}{4}$.

Examination Questions IV. Pp. 146–7.

(1) 15·98 oz. weight. (2) 10 ft. per sec. (3) 7 ; $15\frac{15}{16}$ oz. weight. (5) The one rebounds with a velocity of one foot per second; the other is reduced to rest. (6) 6,400 poundals ; 200 lbs. weight.

Exercises X. Pp. 154–5.

(1) 18,000 ft.-lbs. (2) 12,000 ft.-lbs. ; 5,000 ft.-lbs. (3) 18,000 ft.-lbs. (4) 98·56HP. (5) $\frac{1}{15}$HP. (6) 45. (7) 35,840 ft.-lbs. (8) 198×10^5 ft.-lbs. ; 120HP. (9) 528×10^4 ft.-lbs. (10) A foot-pound $= 1{,}357 \times 10^4$ ergs nearly.

Exercises XI. P. 160.

(1) 462 ft.-lbs. (2) 6,000 ft.-lbs. ; $\frac{1}{3}$. (3) $\mu = \frac{3}{4}$. (4) 4 oz. weight. (5) 3·6 oz. weight. (6) 19,080 ft.-lbs. (7) 6,160 ft.-lbs. ; 560 ft.-lbs. (8) 1·46 oz. weight.

Exercises XII. P. 173.

(1) 960 dynes. (2) 1,080 ft. (3) 3,500 ft.-lbs. (4) 300 ft.-lbs. (5) 80 ft. per sec. nearly. (6) 40$\sqrt{10}$ ft. per sec. (7) 1,444$\frac{44}{100}$ calories. (8) 11,250 ft.-lbs; 14[illegible] units of heat, Fahrenheit.

Examination Questions V. Pp. 180–2.

(1) 40,000 gram-cents.; 12,250 gram-cents.; zero. (2) 1·40 ft. nearly. (3) 1; $m : M$. (4) 87[illegible]. (5) 24,000 g ft.-poundals; 160 ft. per sec., if $g = 32$. (6) 18·2574 ft. per sec.; momentum = 219·0888 ft.-lb.-sec. units; energy = 2,000 ft.-poundals. (7) 107·9 lbs. weight; 1,870 ft.-lbs. (8) [illegible] ton weight; [illegible] ton weight. (9) 200 ft.-lbs.; 150 ft.-lbs. (10) 108. (11) 0·24414 ft. (12) 320. (13) $\frac{w\,l^2}{2}$, where w = weight per foot in lbs., l = length in feet. (14) 11 ft. per sec. (15) 1,500 foot-poundals. (16) 192 ft.-poundals; 5 ft. per sec. (17) 2½ lbs. weight.

Exercises XIII. Pp. 193–4.

(1) 12 ins. from smaller weight. (2) 10 lbs. wt.; 20 lbs. wt. (3) 3 ft. from larger mass. (4) 21·6 ins.; 26 oz. wt. (5) 15 oz. (6) 2 ins. from mass of 6 oz. (7) No. (8) 25 lbs. (9) 18½ lbs. wt. (10) 9·5 lbs.

Exercises XIV. Pp. 199–200.

(1) 16 ins. (2) 8. (3) Increased. (4) 3⅓ ins.

(5) 250 lbs. wt. (6) 560 lbs. (7) The mass of 8 oz. will descend. (8) 240 lbs. (9) $W = n\,t$. (10) 120 lbs. (11) Yes. (12) 90 lbs.

Exercises XV. Pp. 211–4.

(1) 60 lbs. wt. (2) 75 lbs. wt. (3) 20 lbs. wt. (4) 1 lb. wt. (5) 27. (6) 211 oz. wt. (7) 14 lbs. (8) 122 lbs. wt. (9) $3\frac{3}{4}$ ozs. wt. (10) $6 \cdot 341\dot{6}$ lbs. wt. (11) $W = 4P$. (12) 35 lbs. wt. (13) $W = 8P$; force exerted on the beam is $9P$. (14) § 133. (15) A force equal to his weight. (16) 12 lbs. $7\frac{1}{2}$ oz. (17) $17\frac{7}{9}$ lbs. wt.

Exercises XVI. Pp. 222–4.

(1) $6\frac{2}{3}$ oz. (2) $10\frac{1}{2}$ oz. (3) $3\frac{3}{4}$ oz. wt.; $3\frac{3}{7}$ oz. wt.; $8\frac{8}{9}$ oz. wt.; 2 oz. wt. (4) 5 : 12. (5) 5 oz. wt. (6) 12 oz. wt. (7) $(3 + 4\sqrt{2})$ lbs. (8) 7 : 10. (9) $1\frac{3}{7}$ lb. wt. (10) $h : b :: 1 : 2\sqrt{6}$. (11) 2 : 3; 27 ins. (12) 30°. (13) $\dfrac{20}{\sqrt{221}}$ lbs. wt. = 1·34 lb., wt. nearly. (14) $\dfrac{23\sqrt{3}}{120}$. (15) 2 lbs. (16) 2,640 poundals. (17) 12 oz. wt. (18) 100. (19) 1,056 units. (20) $15\frac{5}{9}$ lbs. wt.

Examination Questions VI. Pp. 224–6.

(1) 3 lbs. wt.; the equation of work. (2) As in second system. (3) (a) The first system, five pulleys.; (β) Four pulleys arranged as in third system. (4) $1\frac{69}{176}$ lb. wt. (5) $P = W$. (6) First system, with three pulleys, or second system, with four pulleys in each block; 24 ft. (7) 290 lbs. (8) $12\frac{1}{2}$ ft.-lbs.; $2\frac{1}{2}$ lbs. weight. (9) 20.

(10) 100. (11) 250 lbs. wt.; 2,000 lbs. wt., 1,000 lbs. wt., 500 lbs. wt., 250 lbs. wt. The difference in the sum is the force acting at the free end of the rope. (12) 1,190 lbs. wt.; 1,140 lbs. wt.

EXERCISES XVII. Pp. 231–2.

(1) 4. (2) 12 oz. wt., 9 oz. wt., 5 oz. wt. (3) $8\frac{1}{2}$ ins. (4) 7 cents. or 1 cent., according as the forces act in the same or in opposite directions.

EXERCISES XVIII. Pp. 241–2.

(1) $2\sqrt{31}$. (2) $P\sqrt{5+2\sqrt{2}}$, where P is the smaller force. (3) $P\sqrt{10}$, where P is the smaller force. (4) $P\sqrt{5}$ along a line AM, where M bisects BC. (5) $2\sqrt{61}$ lbs. wt. (6) 17 lbs. wt. (7) 1·2 and 1·6. (8) 2 : 1. (9) $\sqrt{21}$ lbs. wt. (10) 400 lbs. wt. (11) 10. (12) 4 lbs. wt. (13) $2P$, where P is one of the equal forces. (14) $6\sqrt{3}$ poundals. (15) 200; towards the centre. (16) $\sqrt{201} : \sqrt{146} : \sqrt{91}$.

EXERCISES XIX. Pp. 250–2.

(1) No. (2) An equilateral triangle. (3) CB at A. (4) $\frac{10\sqrt{3}}{3}$ oz. wt. (5) $10\sqrt{55+28\sqrt{3}}$ lbs. wt. (6) 33·4 lbs. wt.; 28·8 lbs. wt. (8) $6P$, where P is force along AB. (9) AD. (11) 12 lbs. wt.; 16 lbs. wt. (12) 9 oz.; no weight; 11·1 oz., nearly; 15 oz.

EXERCISES XX. Pp. 259–61.

(1) 1·58 times one of the forces, nearly. (2) 10·2 lbs. wt. (3) $2\sqrt{41}$ lbs. wt. (4) $\sqrt{97+15\sqrt{6}-39\sqrt{2}}$.

(5) (1·67) P, where P is one of the equal forces. (6) 1·8 nearly. (8) 17·32 lbs. wt. (9) 5·95 nearly; 4·91 nearly. (10) 6·928. (12) $P\sqrt{2}$. (14) 52. (15) 122·8 lbs. wt. (16) $P = 28{\cdot}\dot{2}\dot{8}$, where P is one of the equal forces. (17) 7·32 lbs. wt.; 5·17 lbs. wt. (18) 24, nearly.

Exercises XXI. Pp. 277–9.

(1) 2 : 3. (2) $70\frac{7}{12}$ lbs.; $50\frac{5}{12}$ lbs. (3) 20 ins. from end where pressing force is 25 lbs. wt. (4) $6\frac{3}{7}$ ins. from the 5 oz. weight. (5) $\frac{2}{3}$ up median line from angular point. (6) $2\,P\sqrt{2}$, along the diagonal AC. (7) 40 cents. from greater force. (8) 12 ins. (9) $4\frac{7}{9}$ ins. (10) $\frac{5}{8}$ radius. (11) $6\frac{1}{4}$ lbs. wt.; $3\frac{3}{4}$ lbs. wt. (12) $19\frac{1}{3}$ ins. from end where body weighing 4 lbs. is fixed. (13) $2\frac{1}{2}$ lbs. (14) $1\frac{3}{4}$ lbs. (15) $\frac{5}{13}$ of middle line from one of the sides. (16) $7\frac{13}{17}$ ins. from end where force 8 acts. (17) 40 lbs. wt.; 24 lbs. wt. (18) 16; $\frac{7}{16}$ of length of bar from end where force 7 is applied.

Exercises XXII. Pp. 296–7.

(1) $1 : \sqrt{3}$. (2) $\frac{W}{2}.\sqrt{2}$. (3) 1·2 lb. wt. (4) $\frac{4}{5}\sqrt{3}$ lbs. wt. (5) $(8 - 3\sqrt{6}) \times 9{\cdot}6$ ins. from end where weight of 3 lbs. acts. (6) $2\sqrt{3} : 1$. (7) $6\sqrt{2}$ oz. (8) 5 ins. from fulcrum. (9) 5 ins. from end where weight acts. (10) 6 lbs. (11) 12 ins. from end where 4 lbs. is hung; 27 lbs.

Exercises XXIII. Pp. 308–10.

(2) 20 ins.; 40 ins. (3) $2\sqrt{31}$. (4) 6, in direction of force 5. (5) 12·56 ins from end where the force of

4 lbs. weight acts. (6) 10 poundals. (7) If D be the middle point of AB, C the point in the string at which the weight acts, and CE the perpendicular on AB in its position of equilibrium, $\frac{CE}{DE} = \frac{24}{7}$; and when AB is horizontal, the pivot is 6 ins. from D. (8) $(5 + \sqrt{3})$ oz. wt.; $(5 - \sqrt{3})$ oz. wt. (9) $\sqrt{5}$ lbs. wt.; $2\sqrt{5}$ lbs. wt. (10) $16\sqrt{3}$ lbs. wt.; $8\sqrt{3}$ lbs. wt. (11) 12 ins. (12) If C be the nail, B the end of the stick against the wall, $CB = \sqrt[3]{3}$ ft.; the force pressing on the nail is $8\sqrt[3]{3}$ oz. wt., and that on the wall $8\sqrt{\sqrt[3]{9} - 1}$ oz. wt.

Examination Questions VII. Pp. 310–6.

(1) 11·25 lbs. wt.; 27·4 lbs. wt. (2) 15 lbs. wt.; 13 lbs. wt. (7) 6 ft. $4\frac{2}{3}$ ins. from one end. (8) 3 : 2. (9) $\frac{1}{2}$ inch from F. (12) 10 lbs. wt. (14) 0·577 kil. wt. (15) Friction $= 60$; $W = 120\sqrt{3} + 80$; $R = 120\sqrt{3} + 40$. (16) 16 lbs.; 8 inches from the end nearer the point of support. (17) $5\frac{1}{7}$ ins.; $13\frac{1}{7}$ ins.; $29\frac{1}{7}$ ins. and $61\frac{1}{7}$ ins. (18) The resultant equals AB and acts along AM, where M bisects CD. (19) 50 lbs. wt. (20) To a point on the circumference $90° - a$ from the point of contact of the sphere with the plane, a being the angle of the plane. (21) The force on the shoulder varies inversely with distance between hand and shoulder. (22) $\frac{40\sqrt{3}}{3}$ (23) $5 : \sqrt{26} = 1 : 1·0198$. (24) The pull of the string; the weight of the ball; the pull $= \frac{\sqrt{3}}{2}$ lbs. wt. (25) The reaction of the nail and the pull on the string; the pull increases as the string is shortened. (26) The

forces are in equilibrium. (27) 10 lbs. (28) $10\sqrt{2}$ lbs. wt.; 10 lbs. wt. (29) $\frac{1}{6}\sqrt{3}$ ton wt. (30) 0; 380 lbs. wt. (31) 225 lbs. wt.; 135 lbs. wt. (33) 225 lbs. wt., 5 lbs. wt.; 50 lbs. wt., 170 lbs. wt. (34) 34·64 lbs. wt.; 40 lbs. wt. (36) 11·489.

Exercises XXIV. Pp. 340–2.

(2) 240 ft. (3) 48·28 lbs. wt. (4) $\frac{13}{30}$ of length from end where 17 lbs. weight acts. (5) 45°. (6) $\sqrt{8} : 1$. (7) 10 lbs. (8) $\frac{1}{6}$ from centre of square of line joining middle points of AB and CD. (9) $\frac{1}{4}$ diagonal from corner where mass is attached. (10) $\frac{1}{6}$ up median line from base. (11) 4 feet from the other end. (12) If ABC be the triangle and D be a point in AB such that $AD : DB :: 2 : 1$, then mass-centre is the middle point of CD. (13) If a = side of square, the mass-centre is $\frac{5a}{18}$ from centre of square. (14) 8 ins. up median line of larger triangle from the common base. (15) $3\sqrt{3} : 5$.

Exercises XXV. Pp. 353–5.

(1) If A and B be the middle points of the longer and shorter arms, C the angular point, and if $AD = \frac{1}{3} AB$, then mass-centre is $\frac{1}{3}$ up DC. (2) $27\frac{306}{331}$ ft. from end of common axis. (3) $\frac{3a}{2(4+\sqrt{8})}$ from the common side, where a is side of square. (4) $3\frac{2}{3}$ ins. from base. (5) $\frac{1}{6}$ of radius from centre of large circle. (6) $\frac{1}{24}$ of diagonal from centre. (7) The free end is $\frac{2a}{\sqrt{7}}$ verti-

cally below, and $\sqrt{\frac{3}{7}}\,a$ horizontally distant from the point of suspension where a = length of half the wire. (8) If G be the point required, and GF be perpendicular to BC, then $CF = \frac{17}{30}\,a$, and $GF = \frac{7}{15}\,a$ where $a = CB$. (9) 3·464 ins. (10) Reckoning from corner opposite to that from which rectangle is taken $X = 3\frac{8}{17}$ ins, $Y = 2\frac{11}{34}$ ins. (12) 8 : 7. (13) At a point O in AC, where $AO = \dfrac{\sqrt{3}+1}{6}AC$. (14) $\frac{9}{28}$ of line joining the points where masses of 9 kilos. and 3 kilos are placed from mass of 9 kilos. (15) 3·26 ins. of axis from top. (16) $5\frac{4}{4}\frac{}{}$§ ins. from base of shorter cylinder. (17) At centre of sphere.

EXAMINATION QUESTIONS VIII. **Pp. 355–8.**

(1) 18 ins. from end where mass of 1 oz. is placed. (2) $14\frac{1}{21}$ ins. from end of the 8-oz. tube. (3) $\frac{4}{3}\sqrt{3}$ ins. (4) 13·53 ins. (5) $\frac{5}{3}\sqrt{3}$ ins. (6) $2\frac{6}{7}$ ft. from end near the weight of 1 lb. (7) $2\frac{5}{8}$ ft. (8) ·247 in. from centre of the plate. (9) $\frac{1}{6}$ of side of square. (10) §§ 218, 229; the position of equilibrium will remain the same. (11) $7\frac{1}{3}$ ins.; $8\frac{1}{3}$ ins. (12) 12 lbs.; at middle point of rod. (13) 0·316 ft. nearly. (14) $\frac{5}{12}$ up median line to point where mass of 2 lbs. is placed. (15) § 225. If block rest on square base 5×5, $h : b :: 5 : 8$; if on end 8×5, edge 8 being horizontal, $h = b$; if on end 8×5, edge 5 being horizontal, $h : b :: 8 : 5$. (16) $\frac{1}{3}$ weight of plate. (17) 1·45 in. (18) $\frac{1}{48}\,BD$ from centre of square.

MISCELLANEOUS PROBLEMS

1. Draw a diagram, as well as you can to scale, showing the resultant of two forces equal to the weights of 7 and 11 lbs., acting on a particle with an angle of 60° between them; and by measuring the resultant find its numerical value. Indicate two forces at right angles to each other which could maintain equilibrium with the above.—*June*, 1892.
2. A uniform isosceles triangle has its two equal sides each 5 feet long and its base 8 feet long; find its centre of gravity. If its weight be 5 lbs., and a weight of 10 lbs. be hung at the vertex, find the centre of gravity of the whole.—*Ib.*
3. Express the work done when a moment M has rotated n times. If a force equal to the weight of 10 lbs. revolve three times tangentially round a circle of 5 feet radius, find the work it would do. If the energy thus generated were imparted to a free stationary mass of 12 lbs., how fast would it move?—*Ib.*
4. A stone is thrown vertically upwards with a velocity of 160 feet a second. How high will it rise, and how long will it be before it returns to your hand? If you let another stone drop down a well, at the instant the first is within 20 feet of your hand on its return journey, at what distance below your hand will the two bodies meet?—*Ib.*
5. A boy in a toboggan slides down a perfectly smooth hill whose inclination is 1 in 20. At what rate will he be going (in miles per hour) when he has travelled 100 yards from the start?—*Ib.*
6. A cricket ball thrown up is caught by the thrower in 7 seconds. Draw to scale a figure showing its position at the end of every entire second since its start.—*Jan.*, 1893.
7. A uniform bar, 10 feet long, balances over a rail, with a boy weighing three times as much as the bar

hanging on to the extreme end of it. Draw a figure showing its balancing position.—*Ib.*

8. A picture, weighing 56 lbs., is slung over a nail in the ordinary way by a cord attached to two eyes in the top horizontal bar of its frame. If the height of the nail above this bar is half the distance between the eyes, what is the tension of the cord? Under what circumstances would the tension be equal to, or greater than, the whole weight of the picture?—*Ib.*

9. A 3-ton cage descending a shaft with a speed of 9 yards a second is brought to a stop by a uniform force in the space of 18 feet. What is the tension of the rope while the stoppage is occurring? (Express it in tons' weight.)—*Ib.*

10. A steady force applied to a mass of 75 tons, initially moving at a rate of three miles an hour, accelerates it 4 feet a second every second. Calculate (*a*) the applied force in pounds' weight; (*b*) the speed of the body after the lapse of 1·5 minutes; (*c*) its kinetic energy at the same time, expressing it in 'foot-tons.'—*June*, 1893.

11. A stone dropped over a cliff strikes the ground in three seconds. How high is the cliff, and where was the stone when half the time had elapsed? —*Ib.*

12. Calculate the force necessary to hold a hundredweight on a smooth inclined plane tilted 30° from the level, (*a*) if acting horizontally, (*b*) if acting at the best angle.—*Ib.*

13. A couple of unequal weights hang by a thin flexible cord over a perfectly smooth bar. Find the acceleration, the tension on the cord, the distance travelled by either weight from rest in five seconds, and the velocity acquired in the same time; especially for the case when the weights are 49 and 42 grammes respectively.—*Ib.*

14. A stone, weighing 1 ton, is suspended in the air by a chain; a rope fastened to the stone is pulled so that

the chain makes 30° and the rope 60° with the vertical. Draw a very careful figure showing the three forces acting on the stone, and a triangle representing them. Find the pull on the rope.—*Jan.*, 1894.

15. A bar projects 6 inches beyond the edge of a table, and when 2 oz. is hung on to the projecting end the bar just topples over; when it is pushed out so as to project 8 inches beyond the edge, 1 oz. makes it topple over. Find the weight of the bar and the distance of its centre of gravity from the end.—*Ib.*

16. The drum of a windlass is 4 inches in diameter, and the power is applied to the handle 20 inches from the axis. Find the force necessary to sustain the weight of 100 lbs., and the work done in turning the handle 10 times.—*Ib.*

17. A weight of 2 lbs., attached to a string, falls vertically down a mine with uniform acceleration. Find the value of the acceleration if the tension on the string is 1 oz. $(g = 32.)$—*Ib.*

18. A snow slope rises to a height of 50 feet in a slope of 200 feet. A sledge weighing 400 lbs. is drawn up it by a rope parallel to the surface of the snow. Find a triangle representing, in magnitude, the forces acting, and find the pull on the rope when the sledge is going steadily up. Find the work done in pulling the sledge up the slope. Friction is to be neglected.—*June*, 1894.

19. The wheels of a coach are 5 feet apart, and the centre of gravity is 10 feet from the line of contact of the wheels and ground on either side. To what height may the wheels on one side be run up a bank before the coach is upset?—*Ib.*

20. A meteorite burst at a height of 57,600 feet, and one of the fragments was brought instantaneously to rest by the explosion. It then descended with an acceleration of 32 (feet, second), while the sound of the explosion travelled with a velocity of 1,100

feet per second. Which reached the ground first, the fragment or the sound, and what time did each take?—*Ib.*

21. A uniform force equal to the weight of 20 lbs. acts upon a body which is initially at rest, and causes it to move through 24 feet in the first second. Find the mass of the body.—*Ib.*

22. Explain by a diagram how it is possible for a ship to sail partly against the wind.—*Jan.*, 1895.

23. A uniform wire, 10 inches long, is bent so as to consist of two pieces at right angles to each other; one of these pieces is 3 inches long, the other 7 inches. Find the perpendicular distances of the centre of mass of the system from the two portions of the wire.—*Ib.*

24. The average pressure on the piston of a steam engine is 60 lbs. to the square inch, the area of the piston is 1 square foot, and the length of the stroke 18 inches. The engine registers 8 horse-power. How many strokes does it make per minute? (1 horse-power = 33,000 foot-lbs. per minute.)—*Ib.*

25. A uniform force acting on a mass of 6 oz. for 2 seconds generates a velocity of 10 feet per second. Find the measure of the force in dynes. (1 foot = 30·5 cm.; 1 lb. = 453 6 gram.)—*Ib.*

26. A gun weighing 3 kilogrammes fires a bullet weighing 30 grammes, and the latter has a velocity of 60,000 cm. per sec. Compare the kinetic energy of the gun with that of the bullet (neglecting the powder).—*Ib.*

27. A locomotive draws a load of 200 tons. Find the pull needed (1) at constant speed if the friction is ·05 of the load; (2) if the friction is the same, and the speed rises from 30 feet per second to 60 feet per second in one minute. ($g = 32$ feet, second.)—*Ib.*

28. A wire is stretched horizontally between two points 6 feet apart, and a one-pound weight is then hung by a string from the middle of the wire, which is pulled down by the weight 1 inch below the hori-

zontal. Draw a figure showing the forces which act at the point of attachment of the weight to the wire, and find, approximately, the pull on each of the end points to which the wire is fastened.—*June*, 1895.

29. A cubical block of stone, weighing 150 lbs, rests on the ground, which is so rough that the block will not slide. It is to be tilted up so as to rest on one edge. Find the least force which, applied at the opposite edge in a horizontal direction, will begin to tilt the block, and find the least force in any direction at that edge which will begin to tilt it.—*Ib.*

30. A reservoir of water is 11 feet deep and is fed by a stream supplying 60,000 gallons per hour. The water runs out at the same rate at the bottom and turns a turbine. If the turbine uses 60 per cent. of the potential energy which the water loses, find the horse-power which it supplies.—*Ib.*

31. A bird can fly in still air at 20 miles an hour. When a wind is blowing straight from the north at 10 miles an hour, in what direction must the bird aim in order that it may fly from east to west, and at what speed will it travel relative to the earth's surface? Illustrate your answer by a careful figure.—*Ib.*

32. A parachute weighing 1 cwt., falling with a uniform acceleration from rest, descends 16 feet in the first 4 seconds. Find the resultant vertical pressure of the air on the parachute.—*Ib.*

33. A bullet, weighing 1 oz., is fired horizontally from a height of 16 feet. When it strikes the ground the vertical velocity is $\frac{1}{20}$th of the horizontal velocity. Find the energy, in foot-lbs., possessed by the bullet at the instant of projection.—*Ib.*

34. A body, whose mass is 100 lbs., is projected along a horizontal surface with a velocity of 10 feet per second. A constant horizontal retarding force acts on the body, and brings it to rest after the body has passed over 100 feet. Find the magnitude of the retarding force.—*Ib.*

ANSWERS TO MISCELLANEOUS PROBLEMS

(1) 15·7. (2) On central line 2 ft. from vertex; 8 ins. from vertex. (3) 300π foot-lbs.; $v = \sqrt{50 g\pi}$. (4) 400 ft.; 10 secs.; $\frac{5}{16}$ ft. (5) 19·25 miles per hour. (7) Rail is $1\frac{1}{4}$ ft. from boy. (8) The pull on the cord is equal to a weight of $\frac{56}{\sqrt{2}}$ lbs.; *equal*, if angle between cords is 120°; *greater*, if angle is greater. (9) The force required to stop the cage is $\frac{243}{128}$ tons wt. (10) (*a*) 21,000 lbs. wt.; (*b*) 304·4 ft. per. sec.; (*c*) 155,500 foot-tons nearly. (11) 144 ft.; 36 ft. from top. (12) $\frac{112}{\sqrt{3}}$ lbs. wt.; 56 lbs. wt. (13) Acceleration $= \frac{g}{13} =$ 75·5 (cms., sec.) nearly; distance = 944 cms.; vel. = 377 cms. per sec. nearly. (14) 1 ton wt.; an isosceles triangle having vertical angle = 120°. (15) 2 oz.; 12 ins. from end. (16) 10 lbs.; $400\ \pi = 1256{\cdot}6$ foot-lbs. nearly. (17) 31. (18) 100 lbs.; 20,000 foot-lbs. (19) $1\frac{1}{4}$ ft. (20) The fragment takes one minute; the sound $52\frac{4}{11}$ secs. (21) $13\frac{1}{3}$ lbs. (23) 2·45 ins.; 0·45 ins. (24) 20·37 working strokes. (25) 22,950 dynes nearly. (26) 1 : 100. (27) 10 tons wt.; $13\frac{1}{8}$ tons wt. (28) 36 lbs. nearly. (29) 75 lbs. in both cases. (30) 2 HP. (31) 30° to the north of due west; $10\sqrt{3}$ miles per hour. (32) $112 - 7 = 105$ lbs. (33) 400 foot-lbs. (34) 50.

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